Upper functions for positive random functionals. Application to the empirical processes theory II

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Abstract: This part of the paper finalizes the research started in Lepski (2013b).

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1. Introduction

Let us remind the notations and basic assumptions used in Lepski (2013b).

Let $(\mathcal{X}, \mathfrak{X}, \nu)$ be σ-finite space and let $(\Omega, \mathfrak{A}, \mathsf{P})$ be a complete probability space. Let $X_i,\ i \geq 1$, be a the collection of $\mathcal{X}$-valued *independent* random variables defined on $(\Omega, \mathfrak{A}, \mathsf{P})$ and having the densities f_i with respect to measure ν. Furthermore, $\mathbb{P}_{\mathrm{f}},\ \mathrm{f} = (f_1, f_2, \ldots)$, denotes the probability law of $(X_1, X_2, \ldots)$ and $\mathbb{E}_{\mathrm{f}}$ is mathematical expectation with respect to $\mathbb{P}_{\mathrm{f}}$.

Let $G : \boldsymbol{\mathfrak{H}} \times \mathcal{X} \to \mathbb{R}$ be a given mapping, where $\boldsymbol{\mathfrak{H}}$ is a set. Put $\forall n \in \mathbb{N}^*$

$$\xi_{\mathfrak{h}}(n) = n^{-1} \sum_{i=1}^{n} \Big[G(\mathfrak{h}, X_i) - \mathbb{E}_{\mathrm{f}} G(\mathfrak{h}, X_i) \Big], \quad \mathfrak{h} \in \boldsymbol{\mathfrak{H}}. \tag{1.1}$$

We will say that $\xi_{\mathfrak{h}}(n),\ \mathfrak{h} \in \boldsymbol{\mathfrak{H}}$, is a generalized empirical process. Note that if $\mathfrak{h} : \mathcal{X} \to \mathbb{R}$ and $G(\mathfrak{h}, x) = \mathfrak{h}(x),\ \mathfrak{h} \in \boldsymbol{\mathfrak{H}}, x \in \mathcal{X}$, then $\xi_{\mathfrak{h}}(n)$ is the standard empirical process parameterized by $\boldsymbol{\mathfrak{H}}$. We assume that for some $m \geq 1$

$$\boldsymbol{\mathfrak{H}} = \mathfrak{H}_1 \times \cdots \times \mathfrak{H}_m, \tag{1.2}$$

where $\mathfrak{H}_j,\ j = \overline{1, m}$, be given sets. We will use the following notations. For any given $k = \overline{0, m}$ put

$$\mathfrak{H}_1^k = \mathfrak{H}_1 \times \cdots \times \mathfrak{H}_k, \quad \mathfrak{H}_{k+1}^m = \mathfrak{H}_{k+1} \times \cdots \times \mathfrak{H}_m,$$

with the agreement that $\mathfrak{H}_1^0 = \emptyset,\ \mathfrak{H}_{m+1}^m = \emptyset$. The elements of $\mathfrak{H}_1^k$ and $\mathfrak{H}_{k+1}^m$ will be denoted by $\mathfrak{h}^{(k)}$ and $\mathfrak{h}_{(k)}$ respectively. Moreover we suppose that for some $p \geq 1$

$$(\mathcal{X}, \nu) = (\mathcal{X}_1 \times \cdots \times \mathcal{X}_p,\ \nu_1 \times \cdots \times \nu_p), \tag{1.3}$$

where $(\mathcal{X}_l, \nu_l)\ l = \overline{1, p}$, are of measurable spaces and ν is the product measure. It will be also assumed that $\mathfrak{H}_m = \mathcal{X}_1$.

Recall also that all consideration in Lepski (2013b) have been done under the following assumption. Put for any $\mathfrak{h}^{(k)} \in \mathfrak{H}_1^k$

$$\mathbf{G}_\infty(\mathfrak{h}^{(k)}) = \sup_{\mathfrak{h}_{(k)} \in \mathfrak{H}_{k+1}^m} \sup_{x \in \mathcal{X}} |G(\mathfrak{h}, x)|,$$

and let $G_\infty : \mathfrak{H}_1^k \to \mathbb{R}_+$ be any mapping satisfying

$$\mathbf{G}_\infty(\mathfrak{h}^{(k)}) \le G_\infty(\mathfrak{h}^{(k)}), \quad \forall \mathfrak{h}^{(k)} \in \mathfrak{H}_1^k. \tag{1.4}$$

Let $\{\mathfrak{H}_j(n) \subset \mathfrak{H}_j,\ n \ge 1\}, j = \overline{1,k}$, be a sequence of sets and denote $\mathfrak{H}_1^k(n) = \mathfrak{H}_1(n) \times \cdots \mathfrak{H}_k(n)$. Set for any $n \ge 1$

$$\underline{G}_n = \inf_{\mathfrak{h}^{(k)} \in \mathfrak{H}_1^k(n)} G_\infty(\mathfrak{h}^{(k)}), \quad \overline{G}_n = \sup_{\mathfrak{h}^{(k)} \in \mathfrak{H}_1^k(n)} G_\infty(\mathfrak{h}^{(k)}).$$

For any $n \ge 1$, $j = \overline{1,k}$ and any $\mathfrak{h}_j \in \mathfrak{H}_j(n)$ define

$$G_{j,n}(\mathfrak{h}_j) = \sup_{\mathfrak{h}_1 \in \mathfrak{H}_1(n), \ldots, \mathfrak{h}_{j-1} \in \mathfrak{H}_{j-1}(n), \mathfrak{h}_{j+1} \in \mathfrak{H}_{j+1}(n), \ldots, \mathfrak{h}_k \in \mathfrak{H}_k(n)} G_\infty(\mathfrak{h}^{(k)}), \qquad \underline{G}_{j,n} = \inf_{\mathfrak{h}_j \in \mathfrak{H}_j(n)} G_{j,n}(\mathfrak{h}_j).$$

Noting that $\big|\ln(t_1) - \ln(t_2)\big|$ is a metric on $\mathbb{R}_+ \setminus \{0\}$, we equip $\mathfrak{H}_1^k(n)$ with the following semi-metric. For any $n \ge 1$ and any $\hat{\mathfrak{h}}^{(k)}, \bar{\mathfrak{h}}^{(k)} \in \mathfrak{H}_1^k(n)$ set

$$\varrho_n^{(k)}\Big(\hat{\mathfrak{h}}^{(k)}, \bar{\mathfrak{h}}^{(k)}\Big) = \max_{j=\overline{1,k}} \Big|\ln\{G_{j,n}(\hat{\mathfrak{h}}_j)\} - \ln\{G_{j,n}(\bar{\mathfrak{h}}_j)\}\Big|,$$

where $\hat{\mathfrak{h}}_j, \bar{\mathfrak{h}}_j,\ j = \overline{1,k}$, are the coordinates of $\hat{\mathfrak{h}}^{(k)}$ and $\bar{\mathfrak{h}}^{(k)}$ respectively.

Assumption 1. **(i)** $0 < \underline{G}_n \le \overline{G}_n < \infty$ *for any* $n \ge 1$ *and for any* $j = \overline{1,k}$

$$\frac{G_\infty(\mathfrak{h}^{(k)})}{\underline{G}_n} \ge \frac{G_{j,n}(\mathfrak{h}_j)}{\underline{G}_{j,n}}, \quad \forall \mathfrak{h}^{(k)} = (\mathfrak{h}_1, \ldots, \mathfrak{h}_k) \in \mathfrak{H}_1^k(n), \ \forall n \ge 1;$$

(ii) *There exist functions* $L_j : \mathbb{R}_+ \to \mathbb{R}_+,\ D_j : \mathbb{R}_+ \to \mathbb{R}_+,\ j = 0, k+1, \ldots, m$, *satisfying* L_j *non-decreasing and bounded on each bounded interval,* $D_j \in \mathbb{C}^1(\mathbb{R}),\ D(0) = 0$, *and such that*

$$\begin{aligned}\Big\|G(\mathfrak{h}, \cdot) - G(\overline{\mathfrak{h}}, \cdot)\Big\|_\infty &\le \Big\{G_\infty\Big(\mathfrak{h}^{(k)}\Big) \vee G_\infty\Big(\overline{\mathfrak{h}}^{(k)}\Big)\Big\} D_0\Big\{\varrho_n^{(k)}\Big(\mathfrak{h}^{(k)}, \overline{\mathfrak{h}}^{(k)}\Big)\Big\} \\ &+ \sum_{j=k+1}^m L_j\Big\{G_\infty\Big(\mathfrak{h}^{(k)}\Big) \vee G_\infty\Big(\overline{\mathfrak{h}}^{(k)}\Big)\Big\} D_j\Big(\varrho_j(\mathfrak{h}_j, \mathfrak{h}_j')\Big),\end{aligned}$$

for any $\mathfrak{h}, \overline{\mathfrak{h}} \in \mathfrak{H}_1^k(n) \times \mathfrak{H}_{k+1}^m$ *and* $n \ge 1$.

At last the following condition has been also exploited in the previous considerations.

Assumption 2. *For any* $\mathbf{m} \in \mathbb{N}^*$ *there exists* $n[\mathbf{m}] \in \{\mathbf{m}, \mathbf{m}+1, \ldots, 2\mathbf{m}\}$ *such that*

$$\bigcup_{n \in \{\mathbf{m}, \mathbf{m}+1, \ldots, 2\mathbf{m}\}} \widetilde{\mathfrak{H}}_1^k(n) \subseteq \mathfrak{H}_1^k(n[\mathbf{m}]).$$

2. Partially totally bounded case

We begin this section with the following definition used in the sequel. Let $\mathbb{T}$ be a set equipped with a semi-metric $\mathfrak{d}$ and let $\mathfrak{n} \in \mathbb{N}^*$ be fixed.

Definition 1. *We say that* $\{\mathbb{T}_\mathbf{i} \subset \mathbb{T},\ \mathbf{i} \in \mathbf{I}\}$ *is* $\mathfrak{n}$*-totally bounded cover of* $\mathbb{T}$ *if*

- $\mathbb{T} = \cup_{\mathbf{i} \in \mathbf{I}} \mathbb{T}_\mathbf{i}$ *and* $\mathbf{I}$ *is countable;*

- $\mathbb{T}_{\mathbf{i}}$ *is totally bounded for any* $\mathbf{i} \in \mathbf{I}$*;*
- *card*$\Big(\{\mathbf{k} \in \mathbf{I}: \;\; \mathbb{T}_{\mathbf{i}} \cap \mathbb{T}_{\mathbf{k}} \neq \emptyset\}\Big) \leq \mathfrak{n}$ *for any* $\mathbf{i} \in \mathbf{I}$.

Let us illustrate the above definition by some examples.

Let $\mathbb{T} = \mathbb{R}^d, \; d \geq 1$. Then any countable partition of $\mathbb{R}^d$ consisted of bounded sets forms 1-totally bounded cover of $\mathbb{R}^d$. Note, however, that the partitions will not be suitable choice for particular problems studied later. We will be mostly interested in $\mathfrak{n}$-totally bounded covers satisfying the following separation property: there exists $\mathfrak{r} > 0$ such that for all $\mathbf{i}, \mathbf{k} \in \mathbf{I}$ satisfying $\mathbb{T}_{\mathrm{i}} \cap \mathbb{T}_{\mathbf{k}} = \emptyset$

$$\inf_{t_1 \in \mathbb{T}_{\mathrm{i}}, t_2 \in \mathbb{T}_{\mathrm{k}}} \mathfrak{d}(t_1, t_2) \geq \mathfrak{r}. \tag{2.1}$$

Let us return to $\mathbb{R}^d$ that we equip with the metric generated by the supremum norm. Denote by $\mathbb{B}_r(t), \; t \in \mathbb{R}^d, r > 0$, the closed ball in this metric with the radius r and the center t. For given $\mathfrak{r} > 0$ consider the collection $\Big\{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathfrak{r}\mathbf{i}), \; \mathbf{i} \in \mathbb{Z}^d\Big\}$, where we understand $\mathfrak{r}\mathbf{i}$ as coordinate-wise multiplication. It is easy to check that this collection is 3^d-totally bounded cover of $\mathbb{R}^d$ satisfying (2.1).

We would like to emphasize that $\mathfrak{n}$-totally bounded covers satisfying the separation property can be often constructed when $\mathbb{T}$ is a homogenous metric space endowed with the Borel measure obeying doubling condition. Some useful results for this construction can be found in the recent paper Coulhon et al. (2011), where such spaces were scrutinized.

We finish the discussion about $\mathfrak{n}$-totally bounded covers with the following notation: for any $t \in \mathbb{T}$ put

$$\mathbb{T}(t) = \bigcup_{\mathbf{i} \in \mathbf{I}: \; \mathbb{T}_{\mathbf{i}} \ni t} \; \bigcup_{\mathbf{k} \in \mathbf{I}: \; \mathbb{T}_{\mathbf{i}} \cap \mathbb{T}_{\mathbf{k}} \neq \emptyset} \mathbb{T}_{\mathbf{k}}.$$

2.1. Assumptions and main result

Throughout this section we will assume that the representation (1.3) holds and the elements of $\mathcal{X}_l, \; l = \overline{1, p}$, will be denoted by x_l .

Assumption 3. (**i**) *Let (1.2) and (1.3) hold with* $\mathcal{X}_1 = \mathfrak{H}_m$ *and for some* $\mathfrak{n} \in \mathbb{N}^*$ *there exists a collection* $\Big\{\mathrm{H}_{m,\mathbf{i}}, \; \mathbf{i} \in \mathbf{I}\Big\}$ *being the* $\mathfrak{n}$*-totally bounded cover of* $\mathfrak{H}_m$ *satisfying for some* $N, R < \infty$

$$\mathfrak{E}_{\mathrm{H}_{m,\mathbf{i}}, \varrho_m}(\varsigma) \leq N \left[\log_2 \{R/\varsigma\}\right]_+, \quad \forall \mathbf{i} \in \mathbf{I}, \;\; \forall \varsigma > 0.$$

(**ii**) *For any* $\varsigma > 0$

$$\mathfrak{E}_{\mathfrak{H}_j, \varrho_j}(\varsigma) \leq N \left[\log_2 \{R/\varsigma\}\right]_+, \quad \forall j = \overline{k+1, m-1}.$$

Usually one can construct many $\mathfrak{n}$-totally bounded covers satisfying Assumption 3 (**i**). The condition below restricts this choice and relates it to properties of the mapping $G(\cdot, \cdot)$ describing generalized empirical process.

Assumption 4. *For any* $n \geq 1$ *and any* $\mathfrak{h} = (\mathfrak{h}_1, \ldots, \mathfrak{h}_m) \in \mathfrak{H}(n)$

$$\sup_{x \in \mathcal{X}: \; x_1 \notin \mathfrak{H}_m(\mathfrak{h}_m)} |G(\mathfrak{h}, x)| \leq n^{-1} G_\infty(\mathfrak{h}^{(k)}).$$

We would like to emphasize that in order to satisfy Assumption 4 in particular examples, the $\mathfrak{n}$-totally bounded cover $\Big\{\mathrm{H}_{m,\mathbf{i}}, \; \mathbf{i} \in \mathbf{I}\Big\}$ should usually possess the separation property. Indeed, one of the typical examples, where Assumption 4 is fulfilled, is the following: there exist $\gamma > 0$ such that for $G(x, \mathfrak{h}) = 0$ for any $x \in \mathcal{X}, \; \mathfrak{h} \in \mathfrak{H}$, satisfying $\rho_m(x_1, \mathfrak{h}_m) \geq \gamma$.

Result. For any $i=\overline{1,n}$ we denote $X_i=(X_{1,i},\ldots,X_{p,i})$,

$$f_{1,i}(x_1)=\int_{\mathcal{X}_2\times\cdots\times\mathcal{X}_p} f_i(x_1,\ldots,x_p)\prod_{l=2}^{p}\nu_l(\mathrm{d}x_l).$$

and if $\mathcal{X}=\mathcal{X}_1$ $(p=1)$ then we put $X_{1,i}=X_i$ and $f_{1,i}=f_i$.

Put for any $n\geq 1$, $v>0$ and any $\mathfrak{h}_m\in\mathfrak{H}_m$

$$\mathfrak{L}_{n,v}(\mathfrak{h}_m)=-\ln\left(\left[n^{-1}\sum_{i=1}^{n}\int_{\mathfrak{H}_m(\mathfrak{h}_m)} f_{1,i}(x)\nu_1(\mathrm{d}x)\right]\vee n^{-v}\right).$$

Note that obviously $0\leq\mathfrak{L}_{n,v}(\mathfrak{h}_m)\leq v\ln(n),\ \forall\mathfrak{h}_m\in\mathfrak{H}_m$. Put for any $\mathfrak{h}\in\boldsymbol{\mathfrak{H}}$

$$\widetilde{P}(\mathfrak{h})=P(\mathfrak{h}^{(k)})+\mathfrak{L}_{n,v}(\mathfrak{h}_m)+2\ln\{1+|\ln(F_{\mathbf{n_2},\mathbf{r}}(\mathfrak{h}))|\};$$

$$\widetilde{M}_q(\mathfrak{h})=M_q(\mathfrak{h}^{(k)})+\mathfrak{L}_{n,v}(\mathfrak{h}_m)+2\ln\{1+|\ln(F_{\mathbf{n_2},\mathbf{r}}(\mathfrak{h}))|\}.$$

Define for any $\mathfrak{h}\in\boldsymbol{\mathfrak{H}}$, $\mathbf{r}\in\overline{\mathbb{N}}$, $z\geq 0$ and $q>0$

$$\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h})=\lambda_1\sqrt{G_\infty(\mathfrak{h}^{(k)})\Big(F_{\mathbf{n_2},\mathbf{r}}(\mathfrak{h})n^{-1}\Big)\big(\widetilde{P}(\mathfrak{h})+z\big)}+\lambda_2G_\infty(\mathfrak{h}^{(k)})\Big(n^{-1}\ln^{\beta}(n)\Big)\big(\widetilde{P}(\mathfrak{h})+z\big);$$

$$\widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h})=\lambda_1\sqrt{G_\infty(\mathfrak{h}^{(k)})\Big(F_{\mathbf{n_2},\mathbf{r}}(\mathfrak{h})n^{-1}\Big)\big(\widetilde{M}_q(\mathfrak{h})+z\big)}+\lambda_2G_\infty(\mathfrak{h}^{(k)})\Big(n^{-1}\ln^{\beta}(n)\Big)\big(\widetilde{M}_q(\mathfrak{h})+z\big).$$

Theorem 1. *Let Assumptions 1, 3 and 4 hold. If $\mathbf{n_1}\neq\mathbf{n_2}$ suppose additionally that Assumption 2 holds. Then for any $\mathbf{r}\in\mathbb{N}$, $v\geq 1$, $z\geq 1$ and $q\geq 1$*

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}(n)}\Big[|\xi_{\mathfrak{h}}(n)|-\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h})\Big]\geq 0\right\}\leq\mathfrak{n}^5\Big\{4838e^{-z}+4\mathbf{n_1}^{2-v}\Big\};$$

$$\mathbb{E}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}(n)}\Big[|\xi_{\mathfrak{h}}(n)|-\widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h})\Big]\right\}_+^q\leq 2\mathfrak{n}^5c_q\left[\sqrt{(\mathbf{n_1})^{-1}F_{\mathbf{n_2}}\underline{G}_{\mathbf{n}}}\vee\Big((\mathbf{n_1})^{-1}\ln^{\beta}(\mathbf{n_2})\underline{G}_{\mathbf{n}}\Big)\right]^q e^{-z}$$
$$+2^{q+2}\mathfrak{n}^5(\overline{G}_{\mathbf{n}})^q\ \mathbf{n_1}^{2-v}.$$

Although the assertions of the theorem are true whenever $v\geq 1$ the presented results are obviously reasonable only if $v>2$. For example (as we will see later) the typical choice of this parameter for the "moment bound" is $v=q+2$.

In spite of the fact that upper functions presented in Theorem 1 are found explicitly their expressions are quite cumbersome. In particular, it is unclear how to compute the function $\mathfrak{L}_{n,v}(\cdot)$. Of course, since $\mathfrak{L}_{n,v}(\mathfrak{h}_m)\leq v\ln(n),\ \forall\mathfrak{h}_m\in\mathfrak{H}_m$, one can replace it by $v\ln(n)$ in the definition of $\widetilde{P}(\cdot)$ and $\widetilde{M}_q(\cdot)$, but the corresponding upper functions are not always sufficiently tight.

Our goal now is to simplify the expressions for upper functions given in Theorem 1. Surprisingly, that if n is fixed, i.e. $\mathbf{n_1}=\mathbf{n_2}$, it can be done without any additional assumption.

Set for any $v>0$ and $\mathfrak{h}\in\boldsymbol{\mathfrak{H}}$

$$\widehat{P}_v(\mathfrak{h}^{(k)})=P(\mathfrak{h}^{(k)})+2v\left|\ln\Big(2G_\infty(\mathfrak{h}^{(k)})\Big)\right|,\quad\widehat{M}_{q,v}(\mathfrak{h}^{(k)})=M_q(\mathfrak{h}^{(k)})+2v\left|\ln\Big(2G_\infty(\mathfrak{h}^{(k)})\Big)\right|,$$

and let $\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})=\max[F_{\mathbf{n_2}}(\mathfrak{h}),\mathbf{n_2}^{-1}]$.

Corollary 1. *Let the assumptions of Theorem 1 hold. If* $\mathbf{n_1} \neq \mathbf{n_2}$ *suppose additionally that* $X_{i,1},\ i \geq 1$, *are identically distributed.*
Then, the results of Theorem 1 remain valid if one replaces $\widetilde{\mathcal{V}}^{(v,z)}_{\mathbf{r}}(n,\mathfrak{h})$ *and* $\widetilde{\mathcal{U}}^{(v,z,q)}_{\mathbf{r}}(n,\mathfrak{h})$ *by*

$$\begin{aligned}
\widehat{\mathcal{V}}^{(v,z)}(n,\mathfrak{h}) &= \lambda_1\sqrt{G_\infty\big(\mathfrak{h}^{(k)}\big)\Big(\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})n^{-1}\Big)\Big(\widehat{P}_v\big(\mathfrak{h}^{(k)}\big)+2(v+1)\big|\ln\big\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\big\}\big|+z\Big)} \\
&\quad +\lambda_2 G_\infty\big(\mathfrak{h}^{(k)}\big)\Big(n^{-1}\ln^{\beta}(n)\Big)\Big(\widehat{P}_v\big(\mathfrak{h}^{(k)}\big)+2(v+1)\big|\ln\big\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\big\}\big|+z\Big); \\
\widehat{\mathcal{U}}^{(v,z,q)}(n,\mathfrak{h}) &= \lambda_1\sqrt{G_\infty\big(\mathfrak{h}^{(k)}\big)\Big(\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})n^{-1}\Big)\Big(\widehat{M}_{q,v}\big(\mathfrak{h}^{(k)}\big)+2(v+1)\big|\ln\big\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\big\}\big|+z\Big)} \\
&\quad +\lambda_2 G_\infty\big(\mathfrak{h}^{(k)}\big)\Big(n^{-1}\ln^{\beta}(n)\Big)\Big(\widehat{M}_{q,v}\big(\mathfrak{h}^{(k)}\big)+2(v+1)\big|\ln\big\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\big\}\big|+z\Big).
\end{aligned}$$

We would like to emphasize that we do not require that $X_i,\ i \geq 1$, would be identically distributed. In particular, coming back to the generalized empirical process considered in Example 2, Section 1.1 in Lepski (2013b), where $X_i = (Y_i, \varepsilon_i)$, the design points $Y_i,\ i \geq 1$, are often supposed to be uniformly distributed on some bounded domain of $\mathbb{R}^d$. As to the noise variables $\varepsilon_i,\ i \geq 1$, the restriction that they are identically distributed cannot be justified in general.

2.2. Law of logarithm

Our goal here is to use the first assertion of Corollary 1 in order to establish the result referred later to *the law of logarithm*. Namely we show that for some $\boldsymbol{\Upsilon} > 0$

$$\limsup_{n\to\infty}\ \sup_{\mathfrak{h}^{(k)}\in\overline{\mathfrak{H}}_1^k(n,a)} \frac{\sqrt{n}\,\eta_{\mathfrak{h}^{(k)}}(n)}{\sqrt{G_\infty\big(\mathfrak{h}^{(k)}\big)\Big[\ln\big\{G_\infty\big(\mathfrak{h}^{(k)}\big)\big\}\vee\ln\ln(n)\Big]}} \leq \boldsymbol{\Upsilon} \quad \mathbb{P}_{\mathrm{f}}-\text{a.s.} \tag{2.2}$$

As previously we will first provide with the non-asymptotical version of (2.2).

As before we suppose that

$$\mathfrak{c} \leq \underline{G}_n \leq \overline{G}_n \leq \mathfrak{c}n^{\mathfrak{b}}, \quad \forall n \geq 1; \tag{2.3}$$

$$\sup_{n\geq 1}\ \sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}(n)}\ \sup_{i\geq 1}\mathbb{E}_{\mathrm{f}}\big|G(\mathfrak{h},X_i)\big| =: \mathbf{F} < \infty \tag{2.4}$$

and impose additionally the following condition. For some $\mathfrak{a} > 0$

$$\mathcal{L}^{(k)}(z) \leq \mathfrak{a}\ln(z), \quad \forall z \geq 2. \tag{2.5}$$

Theorem 2. *Let Assumptions 1, 2, 3 and 4 be fulfilled. Suppose also that (2.3), (2.4) and (2.5) hold and assume that* $X_{i,1},\ i \geq 1$, *are identically distributed.*

Then there exits $\boldsymbol{\Upsilon} > 0$ *such that for any* $\mathbf{j} \geq 3$ *and any* $a > 4$

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n\geq\mathbf{j}}\ \sup_{\mathfrak{h}^{(k)}\in\overline{\mathfrak{H}}_1^k(n,a)} \frac{\sqrt{n}\,\eta_{\mathfrak{h}^{(k)}}(n)}{\sqrt{G_\infty\big(\mathfrak{h}^{(k)}\big)\Big[\ln\big\{G_\infty\big(\mathfrak{h}^{(k)}\big)\big\}\vee\ln\ln(n)\Big]}} \geq \boldsymbol{\Upsilon}\right\} \leq \frac{4840\mathfrak{n}^5}{\ln(\mathbf{j})}.$$

Some remarks are in order. The explicit expression of the constant $\boldsymbol{\Upsilon}$ is available and the generalization. Also, (2.2) is an obvious consequence of Theorem 2. At last, we note that in view of

(2.3) the factor $\left[\ln\left\{G_\infty(\mathfrak{h}^{(k)})\right\}\vee\ln\ln(n)\right]$ can be replaced by $\ln(n)$ which is, up to a constant, its upper estimate. The corresponding result is, of course, rougher than one presented in the theorem, but its derivation does not require $X_{i,1},\ i\geq 1$, to be identically distributed. This result is deduced directly from Theorem 1. Its proof is almost the same as the proof of Theorem 2 and based on the trivial bound $\mathfrak{L}_{n,v}(\mathfrak{h}_m)\leq v\ln(n),\ \forall\mathfrak{h}_m\in\mathfrak{H}_m$.

3. Application to localized processes

Let $(\mathbb{X}_l,\mu_l,\rho_l),\ l=\overline{1,d+1},\ d\in\mathbb{N}$, be the collection of measurable metric spaces. Throughout this section we will suppose that (1.3) holds with $p=2$,

$$\mathcal{X}=\mathcal{X}_1\times\mathcal{X}_2,\quad (\mathcal{X}_1,\nu_1)=(\mathbb{X}_1\times\cdots\times\mathbb{X}_d,\mu_1\times\cdots\times\mu_d)=:(\mathbb{X}_1^d,\mu^{(d)}),\quad (\mathcal{X}_2,\nu_2)=(\mathbb{X}_{d+1},\mu_{d+1}),$$

x_j denotes the element of $\mathbb{X}_j,\ j=\overline{1,d+1}$, and $x^{(d)}$ will denotes the element of $\mathbb{X}_1^d$. We equip the space $\mathbb{X}_1^d$ with the semi-metric $\rho^{(d)}=\max_{l=\overline{1,d}}\rho_l$.

Problem formulation This section is devoted to the application of Theorem 1 [Lepski (2013b)] and Theorem 1 in the following case:

- $\mathfrak{H}_1^d:=\mathfrak{H}_1\times\cdots\times\mathfrak{H}_d=(0,1]\times\cdots\times(0,1]=(0,1]^d$, (i.e. $k=d$);
- $\mathfrak{H}_{d+1}^{d+2}=\mathfrak{H}_{d+1}\times\mathfrak{H}_{d+2}:=\mathcal{Z}\times\bar{\mathbb{X}}_1^d$, i.e. $m=d+2$, where $\bar{\mathbb{X}}_1^d:=\bar{\mathbb{X}}_1\times\cdots\times\bar{\mathbb{X}}_d$ be a given subset of $\mathbb{X}_1^d$ and $(\mathcal{Z},\mathfrak{d})$ is a given metric space.
- The function $G(\cdot,\cdot)$ obeys some structural assumption described below and for any $\mathfrak{h}:=(r,\mathfrak{z},\bar{x}^{(d)})\in(0,1]^d\times\mathcal{Z}\times\bar{\mathbb{X}}_1^d$ the function $G(\mathfrak{h},\cdot)$ "decrease rapidly " outside of the set $\left\{x_1\in\mathbb{X}_1:\ \ \rho_1(x_1,\bar{x}_1)\leq r_1\right\}\times\cdots\times\left\{x_d\in\mathbb{X}_d:\ \ \rho_d(x_d,\bar{x}_d)\leq r_d\right\}\times\mathbb{X}_{d+1}$.

Let $K:\mathbb{R}^d\to\mathbb{R}$ be a given function, $(\gamma_1,\ldots,\gamma_d)\in\mathbb{R}_+^d$ be given vector and set for any $r\in(0,1]^d$

$$K_r(\cdot)=V_r^{-1}K\left(\cdot/r_1,\ldots,\cdot/r_d\right),\quad V_r=\prod_{l=1}^d r_l^{\gamma_l}.$$

where, as previously, for $u,v\in\mathbb{R}^d$ the notation u/v denotes the coordinate-wise division. Let

$$G(\mathfrak{h},x)=g(\mathfrak{z},x)K_r\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})\right),\quad \mathfrak{h}=\left(r,\mathfrak{z},\bar{x}^{(d)}\right)\in(0,1]^d\times\mathcal{Z}\times\bar{\mathbb{X}}_1^d=:\boldsymbol{\mathfrak{H}},\tag{3.1}$$

where $g:\mathcal{Z}\times\mathcal{X}\to\mathbb{R}$ is a given function those properties will be described later and

$$\vec{\rho}(x^{(d)},\bar{x}^{(d)})=\left(\rho_1(x_1,\bar{x}_1),\ldots,\rho_d(x_d,\bar{x}_d)\right).$$

The corresponding generalized empirical process is given by

$$\xi_{\mathfrak{h}}(n)=n^{-1}\sum_{i=1}^n\left[g(\mathfrak{z},X_i)K_r\left(\vec{\rho}\Big(\left[X_i\right]^{(d)},\bar{x}^{(d)}\Big)\right)-\mathbb{E}_{\mathrm{f}}\left\{g(\mathfrak{z},X_i)K_r\left(\vec{\rho}\Big(\left[X_i\right]^{(d)},\bar{x}^{(d)}\Big)\right)\right\}\right].$$

We will seek upper functions for the random field $\zeta_r(n,\bar{x}^{(d)}):=\sup_{\mathfrak{z}\in\mathcal{Z}}\left|\xi_{r,\mathfrak{z},\bar{x}^{(d)}}(n)\right|$ in two cases: $\bar{\mathbb{X}}_1^d=\mathbb{X}_1^d$ and $\bar{\mathbb{X}}_1^d=\left\{\bar{x}^{(d)}\right\}$ for a fixed $\bar{x}^{(d)}\in\mathbb{X}_1^d$.

To realize this program we will apply Theorem 1 [Lepski (2013b)] and Theorem 1 to $\xi_{\mathfrak{h}}(n)$, $\mathfrak{h} = \left(r, \mathfrak{z}, \bar{x}^{(d)}\right)$. It is worth mentioning that corresponding upper functions can be used for constructing of estimation procedures in different areas of mathematical statistics: M-estimation with locally polynomial fitting (non-parametric regression), kernel density estimation and many others.

Moreover, we apply Theorem 2 [Lepski (2013b)] for establishing a non-asymptotical version of the law of iterated logarithm for $\zeta_r\left(\bar{x}^{(d)}, n\right)$ in the case where $\bar{\mathbb{X}}_1^d = \left\{\bar{x}^{(d)}\right\}$. We also apply Theorem 2 for deriving a non-asymptotical version of the law of logarithm for $\|\zeta_r(n)\|_\infty := \sup_{\bar{x}^{(d)} \in \mathbb{X}_1^d} \left|\zeta_r\left(\bar{x}^{(d)}, n\right)\right|$. Our study here generalizes in several directions the existing results Einmahl and Mason (2000), Giné and Guillou (2002), Einmahl and Mason (2005), Dony et al. (2006), Dony et Einmahl (2009).

3.1. Assumptions and notations

Assumption 5. (**i**) $\|K\|_\infty < \infty$ *and for some* $L_1 > 0$

$$|K(t) - K(s)| \leq \frac{L_1 |t-s|}{1 + |t| \wedge |s|}, \quad \forall t, s \in \mathbb{R}^d,$$

where $|\cdot|$ *denotes supremum norm on* $\mathbb{R}^d$.

$$\text{(ii)}\ \ \|g\|_\infty := \sup_{\mathfrak{z} \in \mathcal{Z},\, x \in \mathcal{X}} |g(\mathfrak{z}, x)| < \infty, \ \textit{and for some}\ \alpha \in (0,1],\ L_\alpha > 0,$$

$$\sup_{x \in \mathcal{X}} |g(\mathfrak{z}, x) - g(\mathfrak{z}', x)| \leq L_\alpha \left[\mathfrak{d}(\mathfrak{z}, \mathfrak{z}')\right]^\alpha, \ \forall \mathfrak{z}, \mathfrak{z}' \in \mathcal{Z};$$

The conditions (**i**) and (**ii**) are quite standards. In particular (**i**) holds if K is compactly supported and lipschitz continuous. If $g(\mathfrak{z}, \cdot) = \bar{g}(\cdot)$, for any $\mathfrak{z} \in \mathcal{Z}$, then (**ii**) is verified for any bounded $\bar{g}$.

Let $0 < r_l^{(\min)}(n) \leq r_l^{(\max)}(n) \leq 1$, $l = \overline{1, d}$, $n \geq 1$, be given decreasing sequences and let

$$\mathfrak{H}(n) = \mathcal{R}(n) \times \mathcal{Z} \times \bar{X}_1^d, \qquad \mathcal{R}(n) = \prod_{l=1}^d \left[r_l^{(\min)}(2n), r_l^{(\max)}(n)\right];$$

$$\widetilde{\mathfrak{H}}(n) = \widetilde{\mathcal{R}}(n) \times \mathcal{Z} \times \bar{X}_1^d, \qquad \widetilde{\mathcal{R}}(n) = \prod_{l=1}^d \left[r_l^{(\min)}(n), r_l^{(\max)}(n)\right].$$

We note that $\widetilde{\mathfrak{H}}(n) \subseteq \mathfrak{H}(n)$ for any $n \geq 1$ since $r_l^{(\min)}(\cdot), r_l^{(\max)}(\cdot)$, $l = \overline{1, d}$, are decreasing, and obviously $\widetilde{\mathfrak{H}}(n) \subseteq \mathfrak{H}(\mathbf{m})$ for any $n \in \{\mathbf{m}, \ldots, \mathbf{2m}\}$ and any $\mathbf{m} \geq 1$.

Remark 1. *Assumption 2 is fulfilled with* $n[\mathbf{m}] = \mathbf{m}$.

Lemma 1. *Suppose that Assumption 5 is fulfilled and let* $\bar{\mathbb{X}}_1^d \subseteq \mathbb{X}_1^d$ *be an arbitrary subset. Then, for arbitrary sequences* $0 < r_l^{(\min)}(n) \leq r_l^{(\max)}(n) \leq 1$, $l = \overline{1, d}$, $n \geq 1$, *Assumption 1 holds with*

$$\varrho_n^{(d)}\left(r, r'\right) = \max_{l = \overline{1,d}} \left|\gamma_l \ln\left(r_l / r_l'\right)\right|, \quad \varrho_{d+1} = \left[\mathfrak{d}\right]^\alpha, \quad \varrho_{d+2} = \max_{l = \overline{1,d}} \rho_l;$$

$$D_0(z) = \exp\{dz\} - 1 + (L_1 / \|K\|_\infty)\left(\exp\left\{\gamma^{-1} z\right\} - 1\right), \quad \gamma = \min_{l = \overline{1,l}} \gamma_l;$$

$$D_{d+1}(z) = (L_\alpha / \|g\|_\infty) z, \quad D_{d+2}(z) = L_1\left(\|g\|_\infty \|K\|_\infty^2\right)^{-1} z, \quad L_{d+1}(z) = z, \quad L_{d+2}(z) = z^2.$$

Additionally, if $\bar{\mathbb{X}}_1^d$ *consists of a single point* $\bar{x}^{(d)} \in \mathbb{X}_1^d$ *then* $L_{d+2} \equiv 0$.

The proof of the lemma is postponed to Appendix. We remark that ϱ_{d+1} is a semi-metric, since $\alpha \in (0,1]$, and the semi-metric $\varrho_n^{(d)}$ is independent on n. In view of latter remark all quantities involved in Assumption 1 are independent on the choice of $r_l^{(\min)}(\cdot), r_l^{(\max)}(\cdot),\ l=\overline{1,d},$. We want to emphasize nevertheless that the assertion of the lemma is true for an arbitrary but *a priory* chosen $r_j^{(\min)}(\cdot), r_l^{(\max)}(\cdot),\ l=\overline{1,d}$.

Thus, Lemma 1 guarantees the verification of Assumption 1, that makes possible the application of Theorem 1 [Lepski (2013b)] and Theorem 1. Hence, we have to match the notations of these theorems to the notations used in the present section.

Since $k=d$ and $\mathfrak{H}_1^k=(0,1]^d$ we have $\mathfrak{h}^{(k)}=r$ and, therefore, in view of Assumption 5

$$\mathbf{G}_\infty(r) := \sup_{(\mathfrak{z},\bar{x}^{(d)})\in\mathcal{Z}\times\bar{\mathbb{X}}_1^d}\ \sup_{x\in\mathbb{X}_1^{d+1}} \left|G\left(\left\{r,\mathfrak{z},\bar{x}^{(d)}\right\},x\right)\right| \le V_r^{-1}\|g\|_\infty\|K\|_\infty =: G_\infty(r).$$

$$\underline{G}_n := \inf_{r\in\mathcal{R}(n)} G_\infty(r) = V_{r^{(\max)}(n)}^{-1}\|g\|_\infty\|K\|_\infty, \quad \forall n\ge 1.$$

We remark that the function $G_\infty(\cdot)$ is independent of the choice of $\bar{\mathbb{X}}_1^d$. Define

$$f_i^{(d)}(x^{(d)}) = \int_{\mathbb{X}_{d+1}} f_i(x)\mu_{d+1}(\mathrm{d}x_{d+1}),\ \ i\ge 1,$$

and let $3\le \mathbf{n_1}\le\mathbf{n_2}\le \mathbf{2n_1}$ be fixed. Set for any $\left(r,\bar{x}^{(d)}\right)\in(0,1]^d\times\bar{\mathbb{X}}_1^d$

$$F_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right) = \begin{cases} \|g\|_\infty(\mathbf{n_2})^{-1}\sum_{i=1}^{\mathbf{n_2}}\int_{\mathbb{X}_1^d}\left|K_r\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})\right)\right| f_i^{(d)}(x^{(d)})\mu^{(d)}(\mathrm{d}x^{(d)}), & \mathbf{n_1}=\mathbf{n_2}; \\ \|g\|_\infty \sup\limits_{i=\overline{1,\mathbf{n_2}}}\int_{\mathbb{X}_1^d}\left|K_r\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})\right)\right| f_i^{(d)}(x^{(d)})\mu^{(d)}(\mathrm{d}x^{(d)}), & \mathbf{n_1}\ne\mathbf{n_2}, \end{cases}$$

and note that in view of Assumption 5 (**ii**)

$$F_{\mathbf{n_2}}(\mathfrak{h}) \le F_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right), \quad \forall\mathfrak{h}\in(0,1]^d\times\mathcal{Z}\times\bar{\mathbb{X}}_1^d.$$

We remark that the function $F_{\mathbf{n_2}}(\cdot,\cdot)$ is independent of the choice of $\mathcal{Z}$. Put also

$$F_{\mathbf{n_2}} := \sup_{n\in\widetilde{\mathbf{N}}}\ \sup_{\left(r,\bar{x}^d\right)\in\mathcal{R}(n)\times\bar{\mathbb{X}}_1^d} F_{\mathbf{n_2}}(r,\bar{x}^d) < \infty,$$

where, remind, $\widetilde{\mathbf{N}}=\{\mathbf{n_1},\dots,\mathbf{n_2}\}$. Finally for any $\mathbf{r}\in\overline{\mathbb{N}}$ set $F_{\mathbf{n_2},\mathbf{r}}(\cdot,\cdot)=\max\left[F_{\mathbf{n_2}}(\cdot,\cdot),e^{-\mathbf{r}}\right]$.

3.2. Pointwise results

Here we will consider the case, where $\bar{\mathbb{X}}_1^d=\left\{\bar{x}^{(d)}\right\}$ and $\bar{x}^{(d)}$ is a fixed element of $\mathbb{X}_1^d$. Note that in view of Lemma 1 $L_{d+1}(z)=z$ and $L_{d+2}\equiv 0$ that implies $\mathcal{L}^{(k)}\equiv 0$.

We are going to apply Theorem 1 from [Lepski (2013b)] and, therefore, we will need the following assumption from Lepski (2013b).

Assumption 6. *Suppose that (1.2) holds and there exist $N,R<\infty$ such that for any $\varsigma>0$ and any $j=\overline{k+1,m}$*

$$\mathfrak{E}_{\mathfrak{H}_j,\varrho_j}(\varsigma)\le N\left[\log_2\{R/\varsigma\}\right]_+,$$

where, as previously, $\mathfrak{E}_{\mathfrak{H}_j,\varrho_j}$ denotes the entropy of $\mathfrak{H}_j$ measured in ϱ_j.

We will suppose that Assumption 6 holds with $k = d, m = d+1$ and $(\mathfrak{H}_{d+1}, \varrho_{d+1}) = (\mathcal{Z}, [\mathfrak{d}]^{\alpha})$. It is equivalent obviously to assume that Assumption 6 holds with $(\mathfrak{H}_{d+1}, \varrho_{d+1}) = (\mathcal{Z}, \mathfrak{d})$ and with the constants $\tilde{N} = \alpha N$ and $\tilde{R} = R^{1/\alpha}$.

Let β and $C_{N,R,m,k}$ be the constants defined in Theorem 1 [Lepski (2013b)]. Set for any $r \in (0,1]^d$ and $q > 0$

$$P(r) = (36d\delta_*^{-2} + 6)\ln\left(1 + \sum_{l=1}^{d} \gamma_l \ln\left\{\frac{2r_l^{(\max)}(n)}{r_l}\right\}\right) + 18C_{N,R,d+1,d};$$

$$M_q(r) = (72d\delta_*^{-2} + 2.5q + 1.5)\sum_{l=1}^{d} \gamma_l \ln\left(\frac{2r_l^{(\max)}(n)}{r_l}\right) + 36C_{N,R,d+1,d}.$$

and define for $\mathbf{r} \in \overline{\mathbb{N}}$ and $u > 0$

$$\begin{aligned}
\mathcal{V}_{\mathbf{r}}^{(u)}(n, r, \bar{x}^d) &= \boldsymbol{\lambda_1}\sqrt{\left[F_{\mathbf{n_2},\mathbf{r}}(r, \bar{x}^d)(nV_r)^{-1}\right]\left[P(r) + 2\ln\left\{1 + \left|\ln\left\{F_{\mathbf{n_2},\mathbf{r}}(r, \bar{x}^d)\right\}\right|\right\} + u\right]} \\
&\quad + \boldsymbol{\lambda_2}\left[(nV_r)^{-1}\ln^{\beta}(n)\right]\left[P(r) + 2\ln\left\{1 + \left|\ln\left\{F_{\mathbf{n_2},\mathbf{r}}(r, \bar{x}^d)\right\}\right|\right\} + u\right]; \\
\mathcal{U}_{\mathbf{r}}^{(u,q)}(n, r, \bar{x}^d) &= \boldsymbol{\lambda_1}\sqrt{\left[F_{\mathbf{n_2},\mathbf{r}}(r, \bar{x}^d)(nV_r)^{-1}\right]\left[M_q(r) + 2\ln\left\{1 + \left|\ln\left\{F_{\mathbf{n_2},\mathbf{r}}(r, \bar{x}^d)\right\}\right|\right\} + u\right]} \\
&\quad + \boldsymbol{\lambda_2}\left[(nV_r)^{-1}\ln^{\beta}(n)\right]\left[M_q(r) + 2\ln\left\{1 + \left|\ln\left\{F_{\mathbf{n_2},\mathbf{r}}(r, \bar{x}^d)\right\}\right|\right\} + u\right],
\end{aligned}$$

where $\boldsymbol{\lambda_1} = \sqrt{\|g\|_\infty \|K\|_\infty}\lambda_1$, $\boldsymbol{\lambda_2} = \|g\|_\infty \|K\|_\infty \lambda_2$ and λ_1, λ_2 are defined in Theorem 1 [Lepski (2013b)].

The result below is the direct consequence of Theorem 1 [Lepski (2013b)] and Lemma 1. We remark that defined above quantities are functions of r and n since $\bar{x}^d$ is fixed. Since they do not depend on the variable $\mathfrak{z}$, these quantities will be automatically upper functions for

$$\zeta_r(n, \bar{x}^{(d)}) := \sup_{\mathfrak{z} \in \mathcal{Z}} \left|\xi_{r,\mathfrak{z},\bar{x}^{(d)}}(\bar{x}^{(d)})\right|.$$

Theorem 3. *Let Assumption 5 be fulfilled and suppose that Assumption 6 holds with $k = d, m = d+1$ and $(\mathfrak{H}_{d+1}, \varrho_{d+1}) = (\mathcal{Z}, [\mathfrak{d}]^{\alpha})$.*

Then for any given decreasing sequences $0 < r_l^{(\min)}(n) \le r_l^{(\max)}(n) \le 1$, $l = \overline{1,d}$, $n \ge 1$, any $\bar{x}^d \in \bar{\mathbb{X}}_1^d$ any $\mathbf{r} \in \mathbb{N}$, $\boldsymbol{b} > 1$ $u \ge 1$ and $q \ge 1$

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n \in \widetilde{\mathbf{N}}} \sup_{r \in \widetilde{\mathcal{R}}(n)} \left[\zeta_r(n, \bar{x}^{(d)}) - \mathcal{V}_{\mathbf{r}}^{(u)}(n, r, \bar{x}^d)\right] \ge 0\right\} \le 2419\, e^{-u};$$

$$\mathbb{E}_{\mathrm{f}}\left\{\sup_{n \in \widetilde{\mathbf{N}}} \sup_{r \in \widetilde{\mathcal{R}}(n)} \left[\zeta_r(n, \bar{x}^{(d)}) - \mathcal{U}_{\mathbf{r}}^{(u,q)}(n, r, \bar{x}^d)\right]\right\}_+^q \le c_q' \left[\sqrt{\frac{F_{\mathbf{n_2}}}{\mathbf{n_1} V_{r^{(\max)}(\mathbf{n_1})}}} \vee \left(\frac{\ln^{\beta}(\mathbf{n_2})}{V_{r^{(\max)}(\mathbf{n_1})}\mathbf{n_1}}\right)\right]^q e^{-u},$$

where $c_q' = 2^{(7q/2)+5}3^{q+4}\Gamma(q+1)\left(C_{D,\boldsymbol{b}}\max\left[\sqrt{\|g\|_\infty \|K\|_\infty}, \|g\|_\infty \|K\|_\infty\right]\right)^q$.

The explicit expression for $C_{D,\boldsymbol{b}}$ can be also found in Theorem 1 [Lepski (2013b)]. In the case considered here it is completely determined by $(\gamma_1, \dots \gamma_d)$, L_1, L_α and $\boldsymbol{b}$.

As well as the assertions of Theorem 1 [Lepski (2013b)] the latter theorem is proved without any assumption imposed on the densities f_i, $i = \overline{1,n}$. The choice of $r_l^{(\min)}(n), r_l^{(\max)}(n)$, $l = \overline{1,d}$, $n \ge 1$, is also assumption free. Additionally, Assumption 6 can be replaced by weaker condition, see [Lepski (2013b)].

Note also that if $g(\mathfrak{z},\cdot) = \bar{g}(\cdot)$, for any $\mathfrak{z} \in \mathcal{Z}$, then Assumption 6 is not needed anymore and, moreover, Assumption 5 (**ii**) is verified for an arbitrary bounded $\bar{g}$. Hence, in this case the assertions of Theorem 3 are established under very mild Assumption 5 (**i**) imposed on the function K.

Remark 2. *We note that the discussed in Introduction so-called price to pay for uniformity disappears if $r = r^{(\max)}$. Indeed, $P\left(r^{(\max)}\right)$ and $M_q\left(r^{(\max)}\right)$ are absolute constants. This property is crucial, in particular, for constructing statistical procedures used in the estimation of functions possessing inhomogeneous smoothness, see Lepski et al. (1997), Kerkyacharian et al. (2001).*

Some additional assumptions and their consequences To apply Theorem 3 to specific problems one needs to find an efficient upper bound for the quantity $F_{\mathbf{n_2}}(\cdot,\cdot)$. Below we provide with sufficient condition allowing to solve this problem under general consideration and we will not be tending here to the maximal generality. We impose some additional restrictions on the densities $f_i,\ i=\overline{1,n}$, and on the measures μ_l of ρ_l-balls in the spaces $\mathbb{X}_l,\ l=\overline{1,d}$. Moreover, we should precise the behavior of the function K at infinity. Then, we will use these assumptions for establishing of the law of iterated logarithm.

Introduce the following notations. For any $t \in \mathbb{R}^d_+$ define

$$\check{K}(t) = \sup_{|u|\notin\Pi_t} |K(u)|, \quad \Pi_t = [0,t_1]\times\cdots\times[0,t_d].$$

For any $l=\overline{1,d},\ x_l \in \mathbb{X}_l$ and $\mathrm{r}>0$ set $\mathbb{B}_l(\mathrm{r},x_l) = \{y \in \mathbb{X}_l :\ \ \rho_l(y,x_l) \le \mathrm{r}\}$.

Assumption 7. *There exists $L_2 > 0$ such that*

$$\sup_{t\in\mathbb{R}^d_+}\left[\Big(\prod_{l=1}^d t_l^{1+\gamma_l}\Big)\check{K}(t)\right] \le L_2; \tag{3.2}$$

For any $l=\overline{1,d}$ and any $x_l \in \mathbb{X}_l$ one has $\mathbb{X}_l = \cup_{\mathrm{r}>0}\left(\mathbb{B}_l(\mathrm{r},x_l)\right)$ and there exist $L^{(l)}>0$

$$\mu_l\Big(\mathbb{B}_l(\mathrm{r},x_l)\Big) \le L^{(l)}\mathrm{r}^{\gamma_l}, \quad \forall \mathrm{r}>0; \tag{3.3}$$

Moreover,

$$\sup_{i\ge1}\sup_{x^{(d)}\in\mathbb{X}_1^d} f_i^{(d)}(x^{(d)}) =: \mathrm{f}_\infty < \infty. \tag{3.4}$$

The condition (3.2) is obviously fulfilled if K is compactly supported on $[0,1]^d$. It is also satisfied in the case of Gaussian or Laplace kernel.

The condition (3.3) can be easily checked if $\mathbb{X}_l,\ l=\overline{1,d}$ are doubling metric spaces. In particular, if $\mathbb{X}_l=\mathbb{R}$ and $\mu_l,\ l=\overline{1,d}$, are the Lebesgue measures than (3.3) holds with $L^{(l)}=1,\ \gamma_l=1,\ l=\overline{1,d}$. If $\mathbb{X}_l=\mathbb{R}^{d_l},\ \ l=\overline{1,d}$, then (3.3) holds with $\gamma_l=d_l$ and the constants $L^{(l)}$ depending on the choice of the distances ρ_l.

As to condition (3.4) we remark that the boundedness of the entire density f_i is not required. For example, under independence structure, i.e. $f_i(x)=f_i^{(d)}(x^{(d)})p_i(x_{d+1})$, the densities p_i may be unbounded.

Lemma 2. *The following bound holds under Assumption 7:*

$$\sup_{\mathbf{n_2}\geq 1}\sup_{r\in(0,1]^d}\sup_{\bar{x}^{(d)}\in\mathbb{X}_1^d} F_{\mathbf{n_2}}\Big(r,\bar{x}^{(d)}\Big)\leq 2^d \mathrm{f}_\infty\|g\|_\infty L_2\prod_{l=1}^{d}2^{\gamma_l}L^{(l)}.$$

The proof of lemma is postponed to Appendix. Our goal now is to deduce the law of iterated logarithm for $\zeta_r(n,\bar{x}^{(d)})$ from Theorem 2, Lepski (2013b). Set for any $n\in\mathbb{N}^*$ and $a>0$

$$\overline{\mathcal{R}}_a(n)=\Big\{r\in(0,1]^d:\ \ V_r\geq\ n^{-1}(\ln n)^a\Big\}.$$

and choose $h^{(\max)}=(1,\dots,1)$ and $h^{(\min)}=(1/n,\dots,1/n)$.

Remark 3. 1^0. *Note that $\overline{\mathcal{R}}_a(n)\subset[n^{-1},1]^d=:\widetilde{\mathcal{R}}(n)$ for any $n\geq 3$ and any $a>0$ and, therefore, the assertion of Lemma 1 holds.*

2^0. *We have $\underline{G}_n=\|K\|_\infty\|g\|_\infty$, $\overline{G}_n=\|K\|_\infty\|g\|_\infty n^{-d}$ for any $n\geq 1$ and, therefore, (2.3) is verified with $\mathfrak{c}=\|K\|_\infty\|g\|_\infty$ and $\mathfrak{b}=d$.*

3^0. *Lemma 2 implies that the condition (2.4) holds with $\mathbf{F}\leq 2^d\mathrm{f}_\infty\|g\|_\infty L_2\prod_{l=1}^{d}2^{\gamma_l}L^{(l)}$.*

4^0. *In view of Lemma 1 $L_{d+1}(z)=z$ and $L_{d+2}\equiv 0$, that implies $\mathcal{L}^{(k)}\equiv 0$. Hence, the condition (1.13) in Lepski (2013b) is fulfilled for any $\mathfrak{a}>0$.*

Thus, all assumptions of Theorem 2, Lepski (2013b), are checked and we come to the following statement.

Theorem 4. *Let Assumptions 5 and 7 be fulfilled and suppose that Assumption 6 holds with $k=d, m=d+1$ and $(\mathfrak{H}_{d+1},\varrho_{d+1})=(\mathcal{Z},[\mathfrak{d}]^\alpha)$. Then there exists $\Upsilon>0$ such that for any $\bar{x}^d\in\bar{\mathbb{X}}_1^d$ and any $a>2$*

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n\geq\mathbf{j}}\ \sup_{r:\,n^{-1}(\ln n)^a\leq V_r\leq 1}\left[\frac{\sqrt{nV_r}\,\zeta_r(n,\bar{x}^{(d)})}{\sqrt{\ln\left(1+\ln(n)\right)}}\right]\geq\Upsilon\right\}\leq\frac{2419}{\ln(\mathbf{j})}.$$

Remark 4. *The inspection of the proof of Theorem 2, Lepski (2013b), together with Lemma 2 allows us to assert that the statement of Theorem 4 is* uniform *over the set of bounded densities.*

More precisely, for any $\mathfrak{f}>0$ there exists $\Upsilon(\mathfrak{f})$ such that

$$\sup_{\mathrm{f}\in\mathcal{F}_{\mathfrak{f}}}\mathbb{P}_{\mathrm{f}}\left\{\sup_{n\geq\mathbf{j}}\ \sup_{r:\,n^{-1}(\ln n)^a\leq V_r\leq 1}\left[\frac{\sqrt{nV_r}\,\zeta_r(n,\bar{x}^{(d)})}{\sqrt{\ln\left(1+\ln(n)\right)}}\right]\geq\Upsilon(\mathfrak{f})\right\}\leq\frac{2419}{\ln(\mathbf{j})},\tag{3.5}$$

where $\mathcal{F}_{\mathfrak{f}}=\big\{(f_i,i\geq 1):\ \mathrm{f}_\infty\leq\mathfrak{f}\big\}$. As before the explicit expression of $\Upsilon(\cdot)$ is available.

The following consequence of Theorem 4 is straightforward.

$$\limsup_{n\to\infty}\ \sup_{r:\,n^{-1}(\ln n)^a\leq V_r\leq 1}\left[\frac{\sqrt{nV_r}\,\zeta_r(n,\bar{x}^{(d)})}{\sqrt{\ln\left(1+\ln(n)\right)}}\right]\leq\Upsilon\quad\mathbb{P}_f-\mathbf{a.s.}\tag{3.6}$$

Theorem 4 generalizes the existing results, see for example Dony et Einmahl (2009), in the following directions.

1. *Structural assumption.* The structural condition (3.1) is imposed in cited papers but with additional restriction: either $g(\mathfrak{z},x) \equiv \text{const}$ ("density case") or $g(\mathfrak{z},x) = \bar{g}(x)$ ("regression case"). It excludes, for instance, the problems appearing in robust estimation. We note that Assumption 5 (**ii**) is fulfilled here if $\bar{g}$ is bounded function and Assumption 6 is not needed anymore, since $\bar{g}$ is independent of $\mathfrak{z}$.
2. *Anisotropy.* All known to the author results treat the case where $\mathbb{X}_l = \mathbb{R},\ l = \overline{1,d}$, and $\mathcal{R}(n) = \left\{(r_1,\ldots,r_d) \in (0,1]^d :\ \ r_l = \mathrm{r},\ \forall l = \overline{1,d},\ \ \mathrm{r} \in \left[\mathrm{r}^{(min)}(n), \mathrm{r}^{(max)}(n)\right]\right\}$ (isotropic case). We remark that (3.3) is automatically fulfilled with $\gamma_l = 1, L^{(l)} = 1,\ l = \overline{1,d}$, and $V_r = \mathrm{r}^d$. Note also that we consider independent but not necessarily identically distributed random variables. This is important, in particular, for various estimation problems arising in nonparametric regression model.
3. *Kernel.* We do not suppose that the function K is compactly supported. For instance, one can use the gaussian or laplace kernel. It allows, for instaince, to consider the problems where X_1^d is not linear space. In particular, it can be some manifold satisfying doubling condition.
4. *Non-asymptotic nature.* The existing results are presented as in (3.6). Note, however, that the random field $\zeta_r(n,\bar{x}^d)$ appears in various areas of nonparametric estimation (density estimation, regression). As the consequence a.s. convergence has no much sense since there is no a unique probability measure (see, also Remark 4).

3.3. Sup-norm results

Here we consider $\bar{\mathbb{X}}_1^d = \mathbb{X}_1^d$. We assume that there exists $\{\mathrm{X}_\mathbf{i},\ \mathbf{i} \in \mathbf{I}\}$ which is $\mathfrak{n}$-totally bounded cover of $\left(\mathbb{X}_1^d, \rho^{(d)}\right)$ satisfying Assumption 3 (**i**) and possessing the separation property.

Assumption 8. *There exists* $\mathfrak{t} > 0$ *such that for any* $\mathbf{i},\mathbf{k} \in \mathbf{I}$ *satisfying* $\mathrm{X}_\mathbf{i} \cap \mathrm{X}_\mathbf{k} = \emptyset$

$$\inf_{x^{(d)} \in \mathrm{X}_\mathbf{i}} \inf_{y^{(d)} \in \mathrm{X}_\mathbf{k}} \rho^{(d)}\left(x^{(d)}, y^{(d)}\right) > \mathfrak{t}.$$

Also we suppose that Assumption 3 (**ii**) holds with $k = d, m = d+1$ and $(\mathfrak{H}_{d+1}, \varrho_{d+1}) = (\mathcal{Z}, [\mathfrak{d}]^\alpha)$. We remark that in the considered case this assumption coincides with Assumption 6.

Let, as previously, $0 < r_l^{(\min)}(n) \le r_l^{(\max)}(n) \le 1,\ l = \overline{1,d},\ n \ge 1$, be given decreasing sequences,

$$\mathfrak{H}(n) = \mathcal{R}(n) \times \mathcal{Z} \times X_1^d, \qquad \mathcal{R}(n) = \prod_{l=1}^d \left[r_l^{(\min)}(2n), r_l^{(\max)}(n)\right];$$
$$\widetilde{\mathfrak{H}}(n) = \widetilde{\mathcal{R}}(n) \times \mathcal{Z} \times X_1^d, \qquad \widetilde{\mathcal{R}}(n) = \prod_{l=1}^d \left[r_l^{(\min)}(n), r_l^{(\max)}(n)\right].$$

Our last condition relates the choice of the vector $r^{(\max)}(n),\ n \ge 1$ and the kernel K with the parameter $\mathfrak{t}$ appearing in Assumption 8. Let us assume that for any $n \ge 1$

$$\sup_{r \in \mathcal{R}(n)} \sup_{|u| \notin (0,\mathfrak{t}]^d} |K(u/r)| \le \|K\|_\infty n^{-1}. \tag{3.7}$$

Note that (3.7) holds if K is compactly supported on $[-\mathfrak{t},\mathfrak{t}]^d$ and $r^{(\max)}(n) \in (0,\mathfrak{t})^d$ for any $n \ge 1$.

Lemma 3. *Assumption 8 and (3.7) imply Assumption 4.*

The proof of lemma is given in Appendix. Set for any $r\in(0,1]^d$ and $v>0$

$$\widehat{M}_{q,v}(r)=\left([72d+108N]\delta_*^{-2}+2.5q+2v+1.5\right)\ln\left(2V_r^{-1}\right)+\mathbf{C},$$

where we have put $\mathbf{C}=72N\delta_*^{-2}\left|\log_2\left(\|g\|_\infty\|K\|_\infty\right)\right|+36C_{N,R,d+1,d}$.

Let $3\leq\mathbf{n_1}\leq\mathbf{n_2}\leq2\mathbf{n_1}$ be fixed. Set $\widehat{F}_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right)=\max\left[F_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right),\mathbf{n_2}^{-1}\right]$ and define

$$\begin{aligned}\widehat{\mathcal{U}}^{(v,z,q)}(n,r,\bar{x}^{(d)}) &= \boldsymbol{\lambda_1}\sqrt{\left[\widehat{F}_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right)(nV_r)^{-1}\right]\left[\widehat{M}_{q,v}(r)+2(v+1)\Big|\ln\left\{\widehat{F}_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right)\right\}\Big|+z\right]}\\ &\quad+\boldsymbol{\lambda_2}\left[(nV_r)^{-1}\ln^{\beta}(n)\right]\left[\widehat{M}_{q,v}(r)+2(v+1)\Big|\ln\left\{\widehat{F}_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right)\right\}\Big|+z\right].\end{aligned}$$

Theorem 5 below is the direct consequence of Lemma 1, Lemma 3 and Corollary 1. Remind that $\zeta_r(n,\bar{x}^{(d)}):=\sup_{x_{d+1}\in\mathbb{X}_{d+1}}\left|\xi_{r,\mathfrak{z},\bar{x}^{(d)}}(\bar{x}^{(d)})\right|$ and $\widetilde{\mathbf{N}}=\{\mathbf{n_1},\dots,\mathbf{n_2}\}$.

Theorem 5. *Let Assumption 5 be verified and suppose that Assumption 3 (**ii**) holds with $k=d+1,m=d+2$ and $(\mathfrak{H}_{d+1},\varrho_{d+1})=(\mathcal{Z},[\mathfrak{d}]^\alpha)$. Suppose also that Assumption 3 (**i**) is fulfilled with $(\mathfrak{H}_{d+2},\varrho_{d+2})=\left(\mathbb{X}_1^d,\rho^{(d)}\right)$ and $\mathrm{H}_{d+2,\mathbf{i}}=\mathrm{X}_{\mathbf{i}},\ \mathbf{i}\in\mathbf{I}$, satisfying Assumption 8. Assume that (3.7) holds as well and if $\mathbf{n_1}\neq\mathbf{n_2}$ let $(X_i)^d$, $i\geq1$, be identically distributed.*

Then for any given decreasing sequences $0<r_l^{(\min)}(n)\leq r_l^{(\max)}(n)\leq1$, $l=\overline{1,d}$, $n\geq1$, any $\boldsymbol{b}>1$, $q\geq1$, $v\geq1$ and $z\geq1$

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\ \sup_{\left(r,\bar{x}^{(d)}\right)\in\widetilde{\mathcal{R}}(n)\times\mathbb{X}_1^d}\left[\zeta_r(n,\bar{x}^{(d)})-\widehat{\mathcal{U}}^{(v,z,q)}(n,r,\bar{x}^{(d)})\right]\geq0\right\}\leq\mathfrak{n}^5\left\{4838e^{-z}+2\mathbf{n_1}^{2-v}\right\};$$

$$\begin{aligned}&\mathbb{E}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\ \sup_{\left(r,\bar{x}^{(d)}\right)\in\widetilde{\mathcal{R}}(n)\times\mathbb{X}_1^d}\left[\zeta_r(n,\bar{x}^{(d)})-\widehat{\mathcal{U}}^{(v,z,q)}(n,r,\bar{x}^{(d)})\right]\right\}_+^q\\ &\quad\leq2\mathfrak{n}^5c_q'\left[\sqrt{\frac{\widehat{F}_{\mathbf{n_2}}}{\mathbf{n_1}V_{r^{(\max)}(\mathbf{n_1})}}}\vee\left(\frac{\ln^{\beta}(\mathbf{n_2})}{V_{r^{(\max)}(\mathbf{n_1})}\mathbf{n_1}}\right)\right]^q e^{-z}+2^{q+1}\mathfrak{n}^5\left(V_{r^{(\min)}(\mathbf{n_1})}\right)^{-q}\mathbf{n_1}^{2-v}.\end{aligned}$$

Remind that $\widehat{F}_{\mathbf{n_2}}=\sup_{n\in\widetilde{\mathbf{N}}}\sup_{\left(r,\bar{x}^{(d)}\right)\in\widetilde{\mathcal{R}}(n)\times\mathbb{X}_1^d}\widehat{F}_{\mathbf{n_2}}\left(r,\bar{x}^{(d)}\right)$ and the expression for the constant c_q' can be found in Theorem 3. We also note that the first assertion of the theorem remains valid if one replaces the quantity $\widehat{M}_{q,v}(r)$ by the smaller quantity $\left([36d+54N]\delta_*^{-2}+2v+6\right)\ln\left(2V_r^{-1}\right)+\mathbf{C}/2$. But the corresponding upper function will differ from $\widehat{\mathcal{U}}^{(v,z,q)}$ only by numerical constant.

We also remark that $\widehat{F}_{\mathbf{n_2}}\leq2^d\mathrm{f}_\infty\|g\|_\infty L_2\prod_{l=1}^d2^{\gamma_l}L^{(l)}$ for any $\mathbf{n_2}\geq3$ under Assumption 7 in view of Lemma 2. Moreover, if $V_{r^{(\min)}(n)}\geq n^{-p}$ for some $p>0$ then $\widehat{M}_{q,v}(r)$ can be bound from above by $([72d+108N]\delta_*^{-2}+2.5q+2v+1.5)p\ln(2n)$ which is independent on r. Hence, if both restrictions are fulfilled the upper function $\widehat{\mathcal{U}}^{(v,z,q)}$ in Theorem 5 takes rather simple form, namely

$$\boldsymbol{\lambda_1}(q)\sqrt{\frac{\ln(n)+z}{nV_r}}+\frac{\boldsymbol{\lambda_2}(q)\left[\ln^{\beta+1}(n)+z\right]}{nV_r},$$

where the constant $\boldsymbol{\lambda_1}(q)$ and $\boldsymbol{\lambda_2}(q)$ can be easily computed.

Law of logarithm In this paragraph we will additionally suppose that Assumption 7 holds. Then, we remark first that statements $1^0 - 3^0$ of Remark 3 hold. Next, we note that $L_{d+1}(z) = z$ and $L_{d+2}(z) = z^2$ in view of Lemma 1 that implies $\mathcal{L}^{(k)}(z) = \ln(z)$ for any $z \geq 1$. Hence, the condition (2.5) is fulfilled with $\mathfrak{a} = 1$.

Thus, all assumptions of Theorem 2 are checked and, taking into account that in our case

$$\eta_{\mathfrak{h}^{(k)}}(n) = \left\|\zeta_r(n)\right\|_\infty := \sup_{\bar{x}^{(d)} \in \mathbb{X}_1^d} \zeta_r\left(n, \bar{x}^{(d)}\right),$$

we come to the following statement.

Theorem 6. *Let assumptions of Theorem 5 be fulfilled and suppose additionally that that Assumption 7 holds. Then there exists* $\boldsymbol{\Upsilon}$ *such that for any* $a > 4$

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n \geq \mathbf{j}} \sup_{r:\, n^{-1}(\ln n)^a \leq V_r \leq 1} \frac{\sqrt{nV_r}\,\left\|\zeta_r(n)\right\|_\infty}{\sqrt{\ln\left(V_r^{-1}\right) \vee \ln\ln(n)}} \geq \boldsymbol{\Upsilon}\right\} \leq \frac{4840\mathfrak{n}^5}{\ln(\mathbf{j})}.$$

The uniform version over the set of bounded densities, similar to (3.5), holds as well.

The immediate consequence of the latter theorem is so-called "*uniform-in-bandwidth consistency*":

$$\limsup_{n\to\infty} \sup_{r:\, n^{-1}(\ln n)^a \leq V_r \leq 1} \frac{\sqrt{nV_r}\,\left\|\zeta_r(n)\right\|_\infty}{\sqrt{\ln\left(V_r^{-1}\right) \vee \ln\ln(n)}} \leq \boldsymbol{\Upsilon} \quad \mathbb{P}_{\mathrm{f}} - \mathbf{a.s} \tag{3.8}$$

The assertion of Theorem 6 and its corollary (3.8) generalizes in several directions the existing results Einmahl and Mason (2000), Giné and Guillou (2002), Einmahl and Mason (2005), Dony et al. (2006) (see, the discussion after Theorem 4).

We would like to conclude this section with the following remark. If K is compactly supported and $g(\mathfrak{z}, \cdot) = \bar{g}(\cdot)$ for any $\mathfrak{z} \in \mathcal{Z}$, where $\bar{g}$ is a bounded function, then all results of this section remain true under Assumptions 3 (**i**), 5 (**i**), 8, (3.3) and (3.4).

4. Proof of Theorems 1–2 and Corollary 1

4.1. Proof of Theorem 1

1^0. We start the proof with establishing some simple facts used in the sequel.

For any $\mathbf{i} \in \mathbf{I}$ let $\mathfrak{n}(\mathbf{i}) \in \mathbb{N}^*$ and $\tilde{\pi}_j(\mathbf{i}) \in \mathbf{I},\ j = 1, \ldots, \mathfrak{n}(\mathbf{i})$, be the pairwise disjoint collection which is determined by the condition: $\mathrm{H}_{m,\mathbf{i}} \cap \mathrm{H}_{m,\mathbf{k}} = \emptyset, \quad \forall \mathbf{k} \notin \left\{\tilde{\pi}_1(\mathbf{i}), \ldots, \tilde{\pi}_{\mathfrak{n}(\mathbf{i})}(\mathbf{i})\right\}$. First we have

$$1 \leq \mathfrak{n}(\mathbf{i}) \leq \mathfrak{n}, \quad \forall \mathbf{i} \in \mathbf{I},$$

and we always put $\tilde{\pi}_{\mathfrak{n}(\mathbf{i})}(\mathbf{i}) = \mathbf{i}$. It yields, in particular, that we can construct another collection of indices $\pi(\mathbf{i}) := \{\pi_j(\mathbf{i}) \in \mathbf{I},\ j = \overline{1, \mathfrak{n}}\}$ given by

$$\pi_j(\mathbf{i}) = \begin{cases} \tilde{\pi}_j(\mathbf{i}), & 1 \leq j \leq \mathfrak{n}(\mathbf{i}); \\ \mathbf{i}, & \mathfrak{n}(\mathbf{i}) + 1 \leq j \leq \mathfrak{n}. \end{cases}$$

Note also that for any $1 \le j \le \mathfrak{n}$

$$\mathrm{card}\Big(\{\mathbf{i} \in \mathbf{I}: \ \ \pi_j(\mathbf{i}) = \mathbf{p}\}\Big) \le \mathfrak{n}, \quad \forall \mathbf{p} \in \mathbf{I}. \tag{4.1}$$

Indeed, if $\mathrm{card}\Big(\{\mathbf{i} \in \mathbf{I}: \ \ \pi_j(\mathbf{i}) = \mathbf{p}\}\Big) \ge \mathfrak{n} + 1$ for some $\mathbf{p} \in \mathbf{I}$, then

$$\mathrm{card}\Big(\{\mathbf{i} \in \mathbf{I}: \ \ \mathrm{H}_{m,\mathbf{p}} \cap \mathrm{H}_{m,\mathbf{i}} \neq \emptyset\}\Big) \ge \mathfrak{n} + 1,$$

that contradicts to the definition of a $\mathfrak{n}$-totally bounded cover. For any $\mathbf{i} \in \mathbf{I}$ define

$$\mathrm{H}_m(\mathbf{i}) = \bigcup_{\mathbf{k} \in \mathbf{I}:\, \mathrm{H}_{m,\mathbf{k}} \cap \mathrm{H}_{m,\mathbf{i}} \neq \emptyset} \ \bigcup_{\mathbf{j} \in \mathbf{I}:\, \mathrm{H}_{m,\mathbf{j}} \cap \mathrm{H}_{m,\mathbf{k}} \neq \emptyset} \mathrm{H}_{m,\mathbf{j}} = \bigcup_{l=1}^{\mathfrak{n}} \bigcup_{j=1}^{\mathfrak{n}} \mathrm{H}_{m,\pi_j\left(\pi_l(\mathbf{i})\right)}.$$

First we note that the definition of the set $\mathfrak{H}_m(\cdot)$ implies the following inclusion: for any $\mathbf{i} \in \mathbf{I}$

$$\mathfrak{H}_m(\mathfrak{h}_m) \subseteq \mathrm{H}_m(\mathbf{i}), \quad \forall \mathfrak{h}_m \in \mathrm{H}_{m,\mathbf{i}}. \tag{4.2}$$

Next, taking into account that $\sum_{\mathbf{q} \in \mathbf{I}} 1_{\mathrm{H}_{m,\mathbf{q}}}(\mathfrak{h}_m) \le \mathfrak{n}$ for any $\mathfrak{h}_m \in \mathfrak{H}_m$ in view of the definition of a $\mathfrak{n}$-totally bounded cover, we obtain in view of (4.1)

$$\begin{aligned}
\sum_{\mathbf{i} \in \mathbf{I}} 1_{\mathrm{H}_m(\mathbf{i})}(\mathfrak{h}_m) &\le \sum_{\mathbf{i} \in \mathbf{I}} \sum_{j=1}^{\mathfrak{n}} \sum_{l=1}^{\mathfrak{n}} 1_{\mathrm{H}_{m,\pi_j\left(\pi_l(\mathbf{i})\right)}}(\mathfrak{h}_m) = \sum_{j=1}^{\mathfrak{n}} \sum_{l=1}^{\mathfrak{n}} \sum_{\mathbf{p} \in \mathbf{I}} \sum_{\mathbf{i}: \pi_l(\mathbf{i}) = \mathbf{p}} 1_{\mathrm{H}_{m,\pi_j(\mathbf{p})}}(\mathfrak{h}_m) \\
&\le \mathfrak{n} \sum_{j=1}^{\mathfrak{n}} \sum_{l=1}^{\mathfrak{n}} \sum_{\mathbf{p} \in \mathbf{I}} 1_{\mathrm{H}_{m,\pi_j(\mathbf{p})}}(\mathfrak{h}_m) = \sum_{j=1}^{\mathfrak{n}} \sum_{l=1}^{\mathfrak{n}} \sum_{\mathbf{q} \in \mathbf{I}} \sum_{\mathbf{p}: \pi_j(\mathbf{p}) = \mathbf{q}} 1_{\mathrm{H}_{m,\mathbf{q}}}(\mathfrak{h}_m) \\
&\le \mathfrak{n}^2 \sum_{j=1}^{\mathfrak{n}} \sum_{l=1}^{\mathfrak{n}} \sum_{\mathbf{q} \in \mathbf{I}} 1_{\mathrm{H}_{m,\mathbf{q}}}(\mathfrak{h}_m) \le \mathfrak{n}^5, \quad \forall \mathfrak{h}_m \in \mathfrak{H}_m.
\end{aligned} \tag{4.3}$$

Define finally for any $\mathbf{i} \in \mathbf{I}$

$$\mathrm{f}_{\mathbf{i}} := \mathbf{n_1}^{-1} \sum_{i=1}^{\mathbf{n_2}} \int_{\mathrm{H}_m(\mathbf{i})} f_{1,i}(x) \nu_1(\mathrm{d}x)$$

and let $\mathbf{I_1} = \{\mathbf{i} \in \mathbf{I}: \ \ \mathrm{f}_{\mathbf{i}} \ge (\mathbf{n_1})^{-v}\}$ and $\mathbf{I_2} = \mathbf{I} \setminus \mathbf{I_1}$.

2^0. Let us fix $\mathbf{i} \in \mathbf{I_1}$ and for any $n \ge 1$ define $\mathrm{H}_{\mathbf{i}}(n) := \widetilde{\mathfrak{H}}_1^k(n) \times \mathfrak{H}_{k+1}^{m-1} \times \mathrm{H}_{m,\mathbf{i}}$, $\mathbf{i} \in \mathbf{I}$. The idea is to apply Theorem 1, Lepski (2013b), to $\{\mathrm{H}_{\mathbf{i}}(n),\ n \ge 1\}$ that is possible in view of Assumptions 3 (**i**) and 2. To do it we first note that the definition of $\mathbf{I_1}$ together with (4.2) implies for any $n \in \widetilde{\mathbf{N}}$

$$\mathfrak{L}_{n,v}(\mathfrak{h}_m) \ge \ln(1/\mathrm{f}_{\mathbf{i}}), \ \ \forall \mathfrak{h}_m \in \mathrm{H}_{m,\mathbf{i}}.$$

It yields for any $n \in \widetilde{\mathbf{N}}$ and any $\mathfrak{h} \in \mathrm{H}_{\mathbf{i}}(n)$

$$\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h}) \ge \mathcal{V}_{\mathbf{r}}^{(u)}(n,\mathfrak{h}), \qquad \widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h}) \ge \mathcal{U}_{\mathbf{r}}^{(u,q)}(n,\mathfrak{h}),$$

where $u = \ln(1/\mathrm{f}_{\mathbf{i}}) + z$. We deduce from Theorem 1, Lepski (2013b), for any $\mathbf{i} \in \mathbf{I_1}$

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n \in \widetilde{\mathbf{N}}} \sup_{\mathfrak{h} \in \mathrm{H}_{\mathbf{i}}(n)} \Big[|\xi_{\mathfrak{h}}(n)| - \widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h})\Big] \ge 0\right\} \le 2419\ \mathrm{f}_{\mathbf{i}}\ e^{-z}; \tag{4.4}$$

$$\mathbb{E}_{\mathrm{f}}\left\{\sup_{n \in \widetilde{\mathbf{N}}} \sup_{\mathfrak{h} \in \mathrm{H}_{\mathbf{i}}(n)} \Big[|\xi_{\mathfrak{h}}(n)| - \widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h})\Big]\right\}_+^q \le \mathrm{f}_{\mathbf{i}}\ \Lambda_q(\mathbf{n_1},\mathbf{n_2}) e^{-z}, \tag{4.5}$$

where we have put $\Lambda_q(\mathbf{n_1},\mathbf{n_2}) = c_q\Big[\sqrt{(\mathbf{n_1})^{-1}F_{\mathbf{n_2}}\underline{G}_{\mathbf{n}}}\vee\Big((\mathbf{n_1})^{-1}\ln^{\beta}(\mathbf{n_2})\underline{G}_{\mathbf{n}}\Big)\Big]^q$.

We have in view of (4.3), taking into account that $\mathbf{n_2}\leq 2\mathbf{n_1}$,

$$\sum_{\mathbf{i}\in\mathbf{I}}\mathrm{f}_{\mathbf{i}} = (\mathbf{n_1})^{-1}\sum_{i=1}^{\mathbf{n_2}}\int f_{1,i}(x)\Big[\sum_{\mathbf{i}\in\mathbf{I}}1_{\mathrm{H}_m(\mathbf{i})}(x)\Big]\nu_1(\mathrm{d}x)\leq 2\mathfrak{n}^5. \tag{4.6}$$

Putting $\widetilde{\mathfrak{H}}^{(1)}(n)=\bigcup_{\mathbf{i}\in\mathbf{I_1}}\mathrm{H}_{\mathbf{i}}(n),\ n\geq 1$, we obtain from (4.4), (4.5) and (4.6)

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\ \sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}^{(1)}(n)}\Big[\big|\xi_{\mathfrak{h}}(n)\big|-\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h})\Big]\geq 0\right\}\leq 4838\ \mathfrak{n}^5\ e^{-z}; \tag{4.7}$$

$$\mathbb{E}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\ \sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}^{(1)}(n)}\Big[\big|\xi_{\mathfrak{h}}(n)\big|-\widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h})\Big]\right\}_+^q\leq 2\Lambda_q(\mathbf{n_1},\mathbf{n_2})\mathfrak{n}^5\ e^{-z}. \tag{4.8}$$

To get (4.8) we have used obvious equality: $[\sup_\alpha Q(\alpha)]_+^q=\sup_\alpha\,[Q(\alpha)]_+^q$.

3^0. Fix $\mathbf{i}\in\mathbf{I_2}$ and note that in view of Assumption 4 for any $n\geq 1$, any $\mathfrak{h}\in\mathrm{H}_{\mathbf{i}}(n)$ and $i\geq 1$

$$\begin{aligned}
\mathbb{E}_{\mathrm{f}}\left|G(\mathfrak{h},X_i)\right| &= \mathbb{E}_{\mathrm{f}}\Big\{\left|G(\mathfrak{h},X_i)\right|\mathbf{1}_{\mathfrak{H}_m(\mathfrak{h}_m)}(X_{1,i})\Big\}+\mathbb{E}_{\mathrm{f}}\Big\{\left|G(\mathfrak{h},X_i)\right|\mathbf{1}_{\mathfrak{H}_m\setminus\mathfrak{H}_m(\mathfrak{h}_m)}(X_{1,i})\Big\}\\
&\leq G_\infty(\mathfrak{h}^{(k)})\Big[\mathbb{P}_{\mathrm{f}}\Big\{X_{1,i}\in\mathfrak{H}_m(\mathfrak{h}_m)\Big\}+n^{-1}\Big]\leq G_\infty(\mathfrak{h}^{(k)})\Big[\mathbb{P}_{\mathrm{f}}\Big\{X_{1,i}\in\mathrm{H}_m(\mathbf{i})\Big\}+n^{-1}\Big].
\end{aligned}\tag{4.9}$$

The last inequality follows from (4.2). It yields for any $n\in\widetilde{\mathbf{N}}$ and any $\mathfrak{h}\in\mathrm{H}_{\mathbf{i}}(n)$

$$n^{-1}\sum_{i=1}^{n}\mathbb{E}_{\mathrm{f}}\left|G(\mathfrak{h},X_i)\right|\ \ \leq\ \ G_\infty(\mathfrak{h}^{(k)})\Big[\mathrm{f}_{\mathbf{i}}+n^{-1}\Big]\leq 2(\mathbf{n_1})^{-1}G_\infty(\mathfrak{h}^{(k)}), \tag{4.10}$$

since $\mathrm{f}_{\mathbf{i}}\leq(\mathbf{n_1})^{-v}$ for any $\mathbf{i}\in\mathbf{I_2}$ and $v\geq 1$.

Introduce random events

$$\mathcal{C}_{\mathbf{i}}=\left\{\sum_{i=1}^{\mathbf{n_2}}\mathbf{1}_{\mathrm{H}_m(\mathbf{i})}(X_{1,i})\geq 2\right\},\ \ \mathbf{i}\in\mathbf{I_2},\qquad \mathcal{C}=\bigcup_{\mathbf{i}\in\mathbf{I_2}}\mathcal{C}_{\mathbf{i}}.$$

Note that if the random event $\bar{\mathcal{C}}$ holds (where, as usual, $\bar{\mathcal{C}}$ is complementary to $\mathcal{C}$) then for any $n\in\widetilde{\mathbf{N}}$ and any $\mathfrak{h}\in\mathrm{H}_{\mathbf{i}}(n)$ in view of Assumption 4 and (4.2)

$$n^{-1}\sum_{i=1}^{n}\left|G(\mathfrak{h},X_i)\right|\ \ \leq\ \ 2n^{-1}G_\infty(\mathfrak{h}^{(k)})\leq 2(\mathbf{n_1})^{-1}G_\infty(\mathfrak{h}^{(k)}). \tag{4.11}$$

Taking into account that bounds found in (4.10) and (4.11) are independent of $\mathbf{i}$ we get for any $n\in\widetilde{\mathbf{N}}$ and any $\mathfrak{h}\in\widetilde{\mathfrak{H}}^{(2)}(n):=\widetilde{\mathfrak{H}}(n)\setminus\in\widetilde{\mathfrak{H}}^{(1)}(n)$

$$\big|\xi_{\mathfrak{h}}(n)\big|\mathbf{1}_{\bar{\mathcal{C}}}\ \ \leq\ \ 4(\mathbf{n_1})^{-1}G_\infty(\mathfrak{h}^{(k)}).$$

Noting that for any $\mathfrak{h}\in\boldsymbol{\mathfrak{H}}$, $z\geq 1$ and $n\in\widetilde{\mathbf{N}}$

$$\begin{aligned}
\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h})\ &>\ 8n^{-1}G_\infty(\mathfrak{h}^{(k)})\geq 4(\mathbf{n_1})^{-1}G_\infty(\mathfrak{h}^{(k)}),\\
\widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h})\ &>\ 8n^{-1}G_\infty(\mathfrak{h}^{(k)})\geq 4(\mathbf{n_1})^{-1}G_\infty(\mathfrak{h}^{(k)}).
\end{aligned}$$

and, therefore, if the random event $\bar{\mathcal{C}}$ is realized we have

$$\sup_{n\in\widetilde{\mathbf{N}}}\sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}^{(2)}(n)}\Big[|\xi_{\mathfrak{h}}(n)|-\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h})\Big]<0,\quad \sup_{n\in\widetilde{\mathbf{N}}}\sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}^{(2)}(n)}\Big[|\xi_{\mathfrak{h}}(n)|-\widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h})\Big]<0.$$

It yields, first,

$$\mathbb{P}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}^{(2)}(n)}\Big[|\xi_{\mathfrak{h}}(n)|-\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(n,\mathfrak{h})\Big]\geq 0\right\}\leq\mathbb{P}_{\mathrm{f}}\left\{\mathcal{C}\right\}\leq\sum_{\mathbf{i}\in\mathbf{I_2}}\mathbb{P}_{\mathrm{f}}\left\{\mathcal{C}_{\mathbf{i}}\right\}. \tag{4.12}$$

Next, taking into account the trivial bound $|\xi_{\mathfrak{h}}(n)|\leq 2\overline{G}_{\mathbf{n}}$ for any $n\in\widetilde{\mathbf{N}}$ and any $\mathfrak{h}\in\mathfrak{H}(n)$, we get

$$\mathbb{E}_{\mathrm{f}}\left\{\sup_{n\in\widetilde{\mathbf{N}}}\sup_{\mathfrak{h}\in\widetilde{\mathfrak{H}}^{(2)}(n)}\Big[|\xi_{\mathfrak{h}}(n)|-\widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(n,\mathfrak{h})\Big]\right\}_+^q\leq\Big(2\overline{G}_{\mathbf{n}}\Big)^q\,\mathbb{P}_{\mathrm{f}}\left\{\mathcal{C}\right\}\leq\Big(2\overline{G}_{\mathbf{n}}\Big)^q\sum_{\mathbf{i}\in\mathbf{I_2}}\mathbb{P}_{\mathrm{f}}\left\{\mathcal{C}_{\mathbf{i}}\right\}. \tag{4.13}$$

For any $\mathbf{i}\in\mathbf{I_2}$ put $\mathrm{p}_{i,\mathbf{i}}=\mathbb{P}_f\Big\{X_{1,i}\in\mathrm{H}_m(\mathbf{i})\Big\}$. Since $X_{1,i},\ i\geq 1$, are independent random elements we have for any $\mathbf{i}\in\mathbf{I_2}$ and any $\lambda>0$ in view of exponential Markov inequality

$$\mathbb{P}_{\mathrm{f}}\left\{\mathcal{C}_{\mathbf{i}}\right\}\leq\exp\left\{-2\lambda+(e^{\lambda}-1)\sum_{i=1}^{\mathbf{n_2}}\mathrm{p}_{i,\mathbf{i}}\right\}=\exp\Big\{-2\lambda+\mathbf{n_1}(e^{\lambda}-1)\mathrm{f}_{\mathbf{i}}\Big\}.$$

Minimizing the right hand side in λ we obtain for any $\mathbf{i}\in\mathbf{I_2}$

$$\mathbb{P}_{\mathrm{f}}\left\{\mathcal{C}_{\mathbf{i}}\right\}\leq(e/2)^2(\mathbf{n_1}\mathrm{f}_{\mathbf{i}})^2\leq 2\mathrm{f}_{\mathbf{i}}\ \mathbf{n_1}^{2-v}.$$

The last inequality follows from the definition of $\mathbf{I_2}$. We obtain finally in view of (4.6)

$$\sum_{\mathbf{i}\in\mathbf{I_2}}\mathbb{P}_{\mathrm{f}}\left\{\mathcal{C}_{\mathbf{i}}\right\}\leq 4\mathfrak{n}^5\ \mathbf{n_1}^{2-v}. \tag{4.14}$$

The assertions of the theorem follow now from (4.7), (4.8), (4.12), (4.13) and (4.14).

∎

4.2. Proof of Corollary 1

To prove the assertion of the corollary it suffices to bound from above the function $\mathfrak{L}_{n,v}(\cdot)$. Remind that we proved, see (4.9), for any $n\geq 1$, any $\mathfrak{h}\in\mathfrak{H}(n)$ and $i\geq 1$

$$\mathbb{E}_{\mathrm{f}}\left|G(\mathfrak{h},X_i)\right|\leq G_\infty(\mathfrak{h}^{(k)})\left[\mathbb{P}_{\mathrm{f}}\Big\{X_{1,i}\in\mathfrak{H}_m(\mathfrak{h}_m)\Big\}+n^{-1}\right].$$

It yields for any $n\in\widetilde{\mathbf{N}}$ and any $\mathfrak{h}\in\mathfrak{H}(n)$

$$F_{\mathbf{n_2}}(\mathfrak{h})\leq G_\infty(\mathfrak{h}^{(k)})\Big[A_n(\mathfrak{h}_m)+n^{-1}\Big],\quad A_n(\mathfrak{h}_m)=n^{-1}\sum_{i=1}^{n}\int_{\mathfrak{H}_m(\mathfrak{h}_m)}f_{1,i}(x)\nu_1(\mathrm{d}x). \tag{4.15}$$

Indeed, if $\mathbf{n_1}=\mathbf{n_2}$ then $n=\mathbf{n_2}$ and (4.15) is obvious. If $\mathbf{n_1}\neq\mathbf{n_2}$ then $\mathbb{P}_{\mathrm{f}}\Big\{X_{1,i}\in\mathfrak{H}_m(\mathfrak{h}_m)\Big\}$ is independent of i since we supposed that $X_{1,i},\ i\geq 1$ are identically distributed. Hence, $A_n(\cdot)$ is independent of n and (4.15) holds. Let $n\in\widetilde{\mathbf{N}}$ be fixed and let $\mathfrak{h}\in\mathfrak{H}(n)$ be such that $F_{\mathbf{n_2}}(\mathfrak{h})\geq n^{-1/2}$.

If $A_n(\mathfrak{h}_m) \le n^{-1}$ we have $G_\infty(\mathfrak{h}^{(k)}) \ge 2^{-1}\sqrt{n}$ and, therefore,

$$2v\Big|\ln\{2G_\infty(\mathfrak{h}^{(k)})\}\Big| \ge v\ln(n) \ge \mathfrak{L}_{n,v}(\mathfrak{h}_m).$$

If $A_n(\mathfrak{h}_m) > n^{-1}$ we have $\widehat{F}_{\mathbf{n_2}}(\mathfrak{h}) = F_{\mathbf{n_2}}(\mathfrak{h}) \le 2G_\infty(\mathfrak{h}^{(k)})A_n(\mathfrak{h}_m)$ and, therefore,

$$\begin{aligned}\mathfrak{L}_{n,v}(\mathfrak{h}_m) \le \ln\Big(A_n^{-1}(\mathfrak{h}_m)\Big) &\le \ln\Big(2G_\infty(\mathfrak{h}^{(k)})\widehat{F}_{\mathbf{n_2}}^{-1}(\mathfrak{h})\Big) = \Big|\ln\Big(2G_\infty(\mathfrak{h}^{(k)})\widehat{F}_{\mathbf{n_2}}^{-1}(\mathfrak{h})\Big)\Big| \\ &\le \Big|\ln\Big(2G_\infty(\mathfrak{h}^{(k)})\Big)\Big| + \Big|\ln\Big(\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\Big)\Big|.\end{aligned}$$

Here we have also used that $A_n(\mathfrak{h}_m) \le 1$. Thus, if $F_{\mathbf{n_2}}(\mathfrak{h}) \ge n^{-1/2}$ for any $v \ge 1$

$$\mathfrak{L}_{n,v}(\mathfrak{h}_m) \le 2v\Big|\ln\Big(2G_\infty(\mathfrak{h}^{(k)})\Big)\Big| + \Big|\ln\Big(\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\Big)\Big|. \tag{4.16}$$

If now $\mathfrak{h} \in \mathfrak{H}(n)$ be such that $F_{\mathbf{n_2}}(\mathfrak{h}) < n^{-1/2}$ then obviously $\widehat{F}_{\mathbf{n_2}}(\mathfrak{h}) < n^{-1/2}$ and, therefore,

$$2v\Big|\ln\Big(\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\Big)\Big| \ge v\ln(n) \ge \mathfrak{L}_{n,v}(\mathfrak{h}_m).$$

The latter inequality together with (4.16) yields for any $n \in \widetilde{\mathbf{N}}$, any $\mathfrak{h} \in \mathfrak{H}(n)$ and $v \ge 1$

$$\mathfrak{L}_{n,v}(\mathfrak{h}_m) \le 2v\Big[\Big|\ln\Big(2G_\infty(\mathfrak{h}^{(k)})\Big)\Big| + \Big|\ln\Big(\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\Big)\Big|\Big]. \tag{4.17}$$

Hence, choosing $\mathbf{r} = \ln(\mathbf{n_2})$ and replacing $\mathfrak{L}_{n,v}(\cdot)$ in the expressions of $\widetilde{\mathcal{V}}_{\mathbf{r}}^{(v,z)}(\cdot,\cdot)$ and $\widetilde{\mathcal{U}}_{\mathbf{r}}^{(v,z,q)}(\cdot,\cdot)$ by its upper bound found in (4.17) we come to the assertion of the corollary.

∎

4.3. Proof of Theorem 2

For any $l \in \mathbb{N}^*$ set $n_l = \mathbf{j}2^l$, $\mathbf{N}_l = \{n_l, n_l+1, \dots, n_{l+1}\}$ and let

$$\zeta_{\mathbf{j}} = \sup_{n\ge\mathbf{j}} \sup_{\mathfrak{h}^{(k)}\in\overline{\mathfrak{H}}_1^k(n,a)} \frac{\sqrt{n}\,\eta_{\mathfrak{h}^{(k)}}(n)}{\sqrt{G_\infty(\mathfrak{h}^{(k)})\Big[\ln\{G_\infty(\mathfrak{h}^{(k)})\} \vee \ln\ln(n)\Big]}}.$$

We obviously have for any $y \ge 0$

$$\begin{aligned}&\mathbb{P}_{\mathrm{f}}\{\zeta_{\mathbf{j}} \ge \boldsymbol{\Upsilon}\} \\ &\le \sum_{l=1}^\infty \mathbb{P}_{\mathrm{f}}\Bigg\{\sup_{n\in\mathbf{N}_l} \sup_{\mathfrak{h}^{(k)}\in\overline{\mathfrak{H}}_1^k(n,a)} \Bigg[\eta_{\mathfrak{h}^{(k)}}(n) - \boldsymbol{\Upsilon}\sqrt{n^{-1}G_\infty(\mathfrak{h}^{(k)})\Big[\ln\{G_\infty(\mathfrak{h}^{(k)})\} \vee \ln\ln(n)\Big]}\Bigg] \ge 0\Bigg\}.\end{aligned}$$

Remind, that for any $3 \le \mathbf{n_1} \le \mathbf{n_2} \le 2\mathbf{n_1}$ and any $n \in \widetilde{\mathbf{N}}$

$$\begin{aligned}\widehat{\mathcal{V}}^{(v,z)}(n,\mathfrak{h}) &= \lambda_1\sqrt{\Big(\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})n^{-1}\Big)G_\infty(\mathfrak{h}^{(k)})\Big(\widehat{P}_v(\mathfrak{h}^{(k)}) + 2(v+1)\big|\ln\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\}\big| + z\Big)} \\ &\quad +\lambda_2\Big(n^{-1}\ln^\beta(n)\Big)G_\infty(\mathfrak{h}^{(k)})\Big(\widehat{P}_v(\mathfrak{h}^{(k)}) + 2(v+1)\big|\ln\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\}\big| + z\Big);\end{aligned}$$

Let $l\in\mathbb{N}^*$ be fixed and choose $v=3$ and $z=2\ln(1+\ln(n_l))$. Later on $\boldsymbol{\Upsilon}_r$, $r=1,2,3,4$ denote the constants independent on l and n.

We have in view of (2.3), (2.4) and (2.5) for any $n\in\mathbf{N}_l$ and $\mathfrak{h}\in\widetilde{\mathfrak{H}}(n)$

$$\widehat{\mathcal{V}}^{(3,\,2\ln(1+\ln(n_l)))}(n,\mathfrak{h})\le\boldsymbol{\Upsilon}_1\sqrt{\frac{G_\infty(\mathfrak{h}^{(k)})\left[\ln\{G_\infty(\mathfrak{h}^{(k)})\}\vee\ln\ln(n)\right]}{n}}+\boldsymbol{\Upsilon}_2\left[\frac{G_\infty(\mathfrak{h}^{(k)})\ln^{\boldsymbol{b}+1}(n)}{n}\right].$$

To get the latter inequality we have used, first, that

$$\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})|\ln\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\}|\le\sup_{x\in(0,\mathbf{F}]}x|\ln(x)|=:c(\mathbf{F})<\infty,\quad\forall\mathbf{F}<\infty.$$

Next, to get the second term, we have used that for any $n\in\mathbf{N}_l$ and $\mathfrak{h}\in\widetilde{\mathfrak{H}}(n)$

$$\widehat{P}_3(\mathfrak{h}^{(k)})\le\boldsymbol{\Upsilon}_3\ln(n),\qquad|\ln\{\widehat{F}_{\mathbf{n_2}}(\mathfrak{h})\}|\le\max\left[|\ln\{\mathbf{F}\}|,\ln(n_{l+1})\right]\le\max\left[|\ln\{\mathbf{F}\}|,\ln(2n)\right].$$

Since $\boldsymbol{b}>1$ can be arbitrary chosen and $a>4$ let $1<\boldsymbol{b}<a/2-1$. It yields for any $n\ge3$ and any $\mathfrak{h}^{(k)}\in\overline{\mathfrak{H}}_1^k(n,a)$

$$\frac{G_\infty(\mathfrak{h}^{(k)})\ln^{\boldsymbol{b}+1}(n)}{n}\le\boldsymbol{\Upsilon}_4\sqrt{\frac{G_\infty(\mathfrak{h}^{(k)})\left[\ln\{G_\infty(\mathfrak{h}^{(k)})\}\vee\ln\ln(n)\right]}{n}}$$

and, therefore, putting $\boldsymbol{\Upsilon}=\boldsymbol{\Upsilon}_1+\boldsymbol{\Upsilon}_2\boldsymbol{\Upsilon}_4$ we get for any $n\in\mathbf{N}_l$

$$\widehat{\mathcal{V}}^{(3,\,2\ln(1+\ln(n_l)))}(n,\mathfrak{h})\le\boldsymbol{\Upsilon}\sqrt{\frac{G_\infty(\mathfrak{h}^{(k)})\left[\ln\{G_\infty(\mathfrak{h}^{(k)})\}\vee\ln\ln(n)\right]}{n}}.$$

Noting that right hand side of the latter inequality is independent of $\mathfrak{h}_{(k)}$ and applying the first assertion of Corollary 1 with $\widetilde{\mathbf{N}}=\mathbf{N}_l$ and $z=2\ln(1+\ln(n_l))$ we obtain

$$\mathbb{P}_{\mathrm{f}}\{\zeta_{\mathbf{j}}\ge\Upsilon\}\le2\mathfrak{n}^5\left\{2419\sum_{l=1}^{\infty}(l+\ln(\mathbf{j}))^{-2}+\mathbf{j}^{-1}\sum_{l=1}^{\infty}2^{-l}\right\}\le2\mathfrak{n}^5\left\{\frac{2419}{\ln(\mathbf{j})}+\mathbf{j}^{-1}\right\}\le\frac{4840\mathfrak{n}^5}{\ln(\mathbf{j})}.$$

∎

5. Appendix

Proof of Lemma 1 1^0. Remind that

$$G_\infty(r)=V_r^{-1}\|g\|_\infty\|K\|_\infty,\qquad\underline{G}_n=V^{-1}_{r^{(\max)}(n)}\|g\|_\infty\|K\|_\infty,\ \ n\ge1.\tag{5.1}$$

Hence we have have for any $l=\overline{1,d}$ and any $\mathfrak{h}_l:=r_l\in\left[r_l^{(\min)}(n),r_l^{(\max)}(n)\right]$

$$G_{l,n}(r_l)=\|g\|_\infty\|K\|_\infty\left[V_{r^{(\min)}(n)}\right]^{-1}\left[r_l^{(\min)}(n)\big/r_l\right]^{\gamma_l};$$

$$\underline{G}_{l,n}=\|g\|_\infty\|K\|_\infty\left[V_{r^{(\min)}(n)}\right]^{-1}\left[r_l^{(\min)}(n)\big/r_l^{(\max)}(n)\right]^{\gamma_l}.$$

Thus, we get for any $n \geq 1$, $r \in \mathcal{R}(n)$ and for any $j = \overline{1,d}$

$$\frac{G_\infty(r)}{\underline{G}_n} = \prod_{l=1}^{d} \left[\frac{r_l^{(\max)}}{r_l}\right]^{\gamma_l} \geq \left[\frac{r_j^{(\max)}}{r_j}\right]^{\gamma_j} = \frac{G_{j,n}(r_j)}{\underline{G}_{j,n}}.$$

We conclude that Assumption 1 (**i**) is fulfilled.

2^0. Remind that for any $r, r' \in \mathcal{R}(n)$

$$\varrho_n^{(d)}\Big(r,r'\Big) := \max_{l=\overline{1,d}} \mathfrak{m}_0\Big(G_{l,n}(r_l), G_{j,n}(r_l')\Big) = \max_{l=\overline{1,d}} \gamma_l \Big|\ln(r_l) - \ln(r_l')\Big| =: \varrho^{(d)}\Big(r,r'\Big), \tag{5.2}$$

3^0. Set $\|K_r - K_{r'}\|_\infty = \sup_{z\in\mathbb{R}^d} |K_r(z) - K_{r'}(z)|$ and note that for any $x \in \mathbb{X}_1^d \times \mathbb{X}_{d+1}$ and for any $\mathfrak{h} = (r, \mathfrak{z}, y^{(d)})$, $\mathfrak{h}' = (r', \mathfrak{z}', z^{(d)})$

$$\begin{aligned}
\big|G(\mathfrak{h},x) - G(\mathfrak{h}',x)\big| &\leq \|g\|_\infty \|K_r - K_{r'}\|_\infty + \|K\|_\infty \left[V_r \vee V_{r'}\right]^{-1} \big|g(\mathfrak{z},x) - g(\mathfrak{z}',x)\big|, \\
&+ \|g\|_\infty V_{r'}^{-1} \Big|K\left(\vec{\rho}(x^{(d)}, y^{(d)})/r'\right) - K\left(\vec{\rho}(x^{(d)}, z^{(d)})/r'\right)\Big| \\
&\leq \|g\|_\infty \|K_r - K_{r'}\|_\infty + L_\alpha \|K\|_\infty \left[V_r \vee V_{r'}\right]^{-1} \left[\mathfrak{d}(\mathfrak{z},\mathfrak{z}')\right]^\alpha \\
&+ \|g\|_\infty V_{r'}^{-1} \Big|K\left(\vec{\rho}(x^{(d)}, y^{(d)})/r'\right) - K\left(\vec{\rho}(x^{(d)}, z^{(d)})/r'\right)\Big|.
\end{aligned}$$

The get the last inequality we have used Assumption 5 (**ii**). Using Assumption 5 (**i**) we have

$$\begin{aligned}
\Big|K\left(\vec{\rho}(x^{(d)}, y^{(d)})/r'\right) - K\left(\vec{\rho}(x^{(d)}, z^{(d)})/r'\right)\Big| &\leq L_1 \max_{l=\overline{1,d}} \Big[(r_l')^{-1} \left|\rho_l(x_l,y_l) - \rho_l(x_l,z_l)\right|\Big] \\
&\leq L_1 \max_{l=\overline{1,d}} \Big[(r_l')^{-1} \rho_l(y_l,z_l)\Big].
\end{aligned}$$

To get the last inequality we have taken into account that ρ_l, $l = \overline{1,d}$, are semi-metrics. Note also that $(r_l')^{-1} \leq V_{r'}^{-1}$ for any $l = \overline{1,d}$, since $r_l' \leq 1$ and we obtain

$$\Big|K\left(\vec{\rho}(x^{(d)}, y^{(d)})/r'\right) - K\left(\vec{\rho}(x^{(d)}, z^{(d)})/r'\right)\Big| \leq L_1 V_{r'}^{-1} \rho^{(d)}(y^{(d)}, z^{(d)}), \tag{5.3}$$

where we have put $\rho^{(d)} = \max_{l=\overline{1,d}} \rho_l$. Obviously,

$$\|K_r - K_{r'}\|_\infty \leq \|K\|_\infty \Big|V_r^{-1} - V_{r'}^{-1}\Big| + \left[V_r \vee V_{r'}\right]^{-1} \left\|K(\,\cdot\,/r) - K(\,\cdot\,/r')\right\|_\infty. \tag{5.4}$$

We have in view of Assumption 5 (**i**) and (5.2)

$$\begin{aligned}
&\left\|K(\,\cdot\,/r) - K(\,\cdot\,/r')\right\|_\infty \leq L_1 \sup_{u\in\mathbb{R}^d} \max_{l=\overline{1,d}} \left[\frac{|u_l|\,|1/r_l - 1/r_l'|}{1 + |u_l|\,(1/r_l \wedge 1/r_l')}\right] \leq L_1 \max_{l=\overline{1,d}} \left[\frac{r_l \vee r_l'}{r_l \wedge r_l'} - 1\right] \\
&= L_1\Big[\exp\Big\{\max_{l=\overline{1,d}} \Big|\ln(r_l) - \ln(r_l')\Big|\Big\} - 1\Big] \leq L_1\Big[\exp\Big\{\gamma^{-1}\varrho^{(d)}\Big(r,r'\Big)\Big\} - 1\Big],
\end{aligned}$$

we have put $\gamma = \min[\gamma_1, \ldots, \gamma_d]$. Moreover, we obviously have for any $r, r' \in (0,1]^d$

$$\frac{V_r \vee V_{r'}}{V_r \wedge V_{r'}} \leq \frac{V_{r\vee r'}}{V_{r\wedge r'}} = \exp\left\{\sum_{l=1}^{d} \gamma_l \Big|\ln(r_l) - \ln(r_l')\Big|\right\} \leq \exp\Big\{d\varrho^{(d)}(r,r')\Big\}.$$

Thus, we finally obtain from (5.4)

$$\|K_r - K_{r'}\|_\infty \le [V_r \vee V_{r'}]^{-1}\Big[\|K\|_\infty\Big(\exp\Big\{d\varrho^{(d)}(r,r')\Big\}-1\Big)+L_1\Big[\exp\Big\{\gamma^{-1}\varrho^{(d)}\Big(r,r'\Big)\Big\}-1\Big]\Big].$$

This yields together with (5.3) for any $\mathfrak{h}=(r,\mathfrak{z},y^{(d)})$ and $\mathfrak{h}'=(r',\mathfrak{z}',z^{(d)})$

$$\begin{aligned}
&\sup_{x\in\mathbb{X}_1^d\times\mathbb{X}_{d+1}}\big|G(\mathfrak{h},x)-G(\mathfrak{h}',x)\big| \qquad (5.5)\\
&\le \|g\|_\infty[V_r\vee V_{r'}]^{-1}\Big[\|K\|_\infty\Big(\exp\Big\{d\varrho^{(d)}(r,r')\Big\}-1\Big)+L_1\Big[\exp\Big\{\gamma^{-1}\varrho^{(d)}\Big(r,r'\Big)\Big\}-1\Big]\Big]\\
&\qquad +L_\alpha\|K\|_\infty[V_r\vee V_{r'}]^{-1}\left[\mathfrak{d}(\mathfrak{z},\mathfrak{z}')\right]^\alpha+L_1\|g\|_\infty V_{r'}^{-2}\rho^{(d)}(y^{(d)},z^{(d)})\\
&\le \|g\|_\infty\|K\|_\infty[V_r\wedge V_{r'}]^{-1}\Big[D_0\Big(\varrho^{(d)}\Big)+D_{d+1}(\varrho_{d+1})+[V_r\wedge V_{r'}]^{-1}D_{d+2}\Big(\rho^{(d)}(y^{(d)},z^{(d)})\Big)\Big],
\end{aligned}$$

where we have put $\varrho_{d+1}=[\mathfrak{d}]^\alpha$, $D_{d+1}(z)=(L_\alpha/\|g\|_\infty)z$, $D_{d+2}(z)=L_1(\|g\|_\infty\|K\|_\infty^2)^{-1}z$ and

$$D_0(z)=\exp\{dz\}-1+(L_1/\|K\|_\infty)\Big(\exp\Big\{\gamma^{-1}z\Big\}-1\Big).$$

Putting $L_{d+1}(z)=z$ and $L_{d+2}(z)=z^2$ we obtain from (5.1) and (5.5) for any $\mathfrak{h}=(r,\mathfrak{z},y^{(d)})$ and $\mathfrak{h}'=(r',\mathfrak{z}',z^{(d)})$

$$\begin{aligned}
&\sup_{x\in\mathbb{X}_1^d\times\mathbb{X}_{d+1}}\big|G(\mathfrak{h},x)-G(\mathfrak{h}',x)\big|\le G_\infty(r)\vee G_\infty(r')D_0\Big(\varrho^{(d)}(r,r')\Big)\\
&+L_{d+1}\left(G_\infty(r)\vee G_\infty(r')\right)D_{d+1}\Big(\varrho_{d+1}(\mathfrak{z},\mathfrak{z}')\Big)+L_{d+2}\left(G_\infty(r)\vee G_\infty(r')\right)D_{d+2}\left(\rho^{(d)}(y^{(d)},z^{(d)})\right).
\end{aligned}$$

We conclude that Assumption 1 (**ii**) is fulfilled. It remains to note that if $\bar{X}_1^d$ consists of a single element then last summand in the right hand side of the latter inequality disappears that correspond formally to $L_{d+2}\equiv 0$. This completes the proof of the lemma.

∎

Proof of Lemma 2 In view of (3.4) for any $r\in(0,1]^d$

$$F_{\mathbf{n_2}}\Big(r,\bar{x}^{(d)}\Big)\le \mathrm{f}_\infty\|g\|_\infty\int_{\mathbb{X}_1^d}\Big|K_r\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})\right)\Big|\mu^{(d)}(\mathrm{d}x^{(d)})=:\mathrm{f}_\infty\|g\|_\infty\mathcal{I}_r. \qquad (5.6)$$

Denote for any $l=\overline{1,d}$

$$\mathfrak{R}_l(k_l,r_l)=\mathbb{B}_l\Big(2^{k_l+1}r_l,\bar{x}_l\Big)\setminus\mathbb{B}_l\Big(2^{k_l}r_l,\bar{x}_l\Big),\quad \mathfrak{R}_l(0,r_l)=\mathbb{B}_l\Big(r_l,\bar{x}_l\Big),\quad k_l\in\mathbb{N}.$$

and for any multi-index $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{N}^d$ set $\mathfrak{R}_{\mathbf{k},r}=\mathfrak{R}_1(k_1,r_1)\times\cdots\times\mathfrak{R}_d(k_d,r_d)$. We get in view of Assumption 7 that $\mathbb{X}_1^d=\bigcup_{\mathbf{k}\in\mathbb{N}^d}\mathfrak{R}_{\mathbf{k},r}$ for any $r\in(0,1]^d$ and, therefore,

$$\mathcal{I}_r=\sum_{\mathbf{k}\in\mathbb{N}^d}\int_{\mathfrak{R}_{\mathbf{k},r}}\Big|K_r\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})\right)\Big|\mu^{(d)}(\mathrm{d}x^{(d)}).$$

We note that for any $\mathbf{k}\in\mathbb{N}^d$ that for any $x^{(d)}\in\mathfrak{R}_{\mathbf{k},r}$

$$\left|K_r\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})\right)\right| = V_r^{-1}\left|K\left(\frac{\vec{\rho}(x^{(d)},\bar{x}^{(d)})}{r}\right)\right| \le V_r^{-1}\sup_{|u|\notin\Pi_{\mathbf{t}(\mathbf{k})}}|K(u)| = V_r^{-1}\check{K}(\mathbf{t}(\mathbf{k})).$$

where, we have put $\mathbf{t}(\mathbf{k}) = (2^{k_1},\ldots,2^{k_d})$ and where, remind, $\Pi_t = [0,t_1]\times\cdots\times[0,t_d],\ t\in\mathbb{R}_+^d$.

Thus, we obtain from (3.3) of Assumption 7 (remind that $\mu^{(d)}$ is a product measure)

$$\begin{aligned}\mathcal{I}_r &\le V_r^{-1}\sum_{\mathbf{k}\in\mathbb{N}^d}\check{K}(\mathbf{t}(\mathbf{k}))\mu^{(d)}\left(\Pi_{\mathbf{t}(\mathbf{k})}\right) \le V_r^{-1}\sum_{\mathbf{k}\in\mathbb{N}^d}\check{K}(\mathbf{t}(\mathbf{k}))\left[\prod_{l=1}^d \mu_l\left(\mathbb{B}_l(2^{k_l+1}r_l,\bar{x}_l)\right)\right]\\ &\le \left[\prod_{l=1}^d 2^{\gamma_l}L^{(l)}\right]\sum_{\mathbf{k}\in\mathbb{N}^d}\check{K}(\mathbf{t}(\mathbf{k}))\left[\prod_{l=1}^d 2^{\gamma_l k_l}\right]. \end{aligned} \tag{5.7}$$

We get finally from (3.2) of Assumption 7 that for any $r\in(0,1]^d$

$$\mathcal{I}_r \le 2^d L_2\prod_{l=1}^d 2^{\gamma_l}L^{(l)}.$$

The assertion of the lemma follows now from (5.6).

∎

Proof of Lemma 3 Remind, that for the considered problem

$$\mathfrak{H}(\mathfrak{h}_{d+2}) = \mathbb{X}_1^d(\bar{x}^{(d)}) := \bigcup_{\mathbf{i}:\ \bar{x}^{(d)}\in\mathrm{X}_\mathbf{i}}\ \bigcup_{\mathbf{k}:\ \mathrm{X}_\mathbf{k}\cap\mathrm{X}_\mathbf{i}\neq\emptyset}\mathrm{X}_\mathbf{k}.$$

For any $\bar{x}^{(d)}\in\mathbb{X}_1^d$ and any $\mathrm{r}>0$ denote $\mathbb{B}_{\rho^{(d)}}(\mathrm{r},\bar{x}^{(d)}) = \left\{x^{(d)}\in\mathbb{X}_1^d:\ \ \rho^{(d)}(x^{(d)},\bar{x}^{(d)})\le\mathrm{r}\right\}$ where, remind, $\rho^{(d)} = \max[\rho_1,\ldots,\rho_d]$. The following inclusion holds in view of Assumption 8

$$\mathbb{B}_{\rho^{(d)}}(\mathfrak{t},\bar{x}^{(d)}) \subseteq \mathbb{X}_1^d(\bar{x}^{(d)}),\ \ \forall\bar{x}^{(d)}\in\mathbb{X}_1^d. \tag{5.8}$$

Indeed, suppose that $\exists y^{(d)}\in\mathbb{B}_{\rho^{(d)}}(\mathfrak{t},\bar{x}^{(d)})$ such that $y^{(d)}\notin\mathbb{X}_1^d(\bar{x}^{(d)})$. Then, the definition of $\mathbb{X}_1^d(\bar{x}^{(d)})$ implies that for any $\mathbf{p},\mathbf{q}\in\mathbf{I}$ such that $\bar{x}^{(d)}\in\mathrm{X}_\mathbf{p},\ \ y^{(d)}\in\mathrm{X}_\mathbf{q}$ necessarily

$$\mathrm{X}_\mathbf{p}\cap\mathrm{X}_\mathbf{q}=\emptyset.$$

Hence, in view of Assumption 8, $\rho^{(d)}(y^{(d)},\bar{x}^{(d)})>\mathfrak{t}$ and, therefore, $y^{(d)}\notin\mathbb{B}_{\rho^{(d)}}(\mathfrak{t},\bar{x}^{(d)})$. The obtained contradiction proves (5.8).

Note that in view of Assumption 5 (**ii**) for any $x\in\mathbb{X}_1^d\times\mathbb{X}_{d+1}$ and any $\mathfrak{h}=(r,\mathfrak{z},\bar{x}^{(d)})$

$$|G(\mathfrak{h},x)| \le \|g\|_\infty V_r^{-1}\left|K\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})/r\right)\right|$$

and, therefore, we get from (5.8) and (3.7)

$$\begin{aligned}\sup_{x\in\mathbb{X}_1^d\times\mathbb{X}_{d+1}:\ x^{(d)}\notin\mathbb{X}_1^d(\bar{x}^{(d)})}|G(\mathfrak{h},x)| &\le \|g\|_\infty V_r^{-1}\sup_{x^{(d)}\notin\mathbb{B}_{\rho^{(d)}}(\mathfrak{t},\bar{x}^{(d)})}\left|K\left(\vec{\rho}(x^{(d)},\bar{x}^{(d)})/r\right)\right|\\ &\le \|g\|_\infty V_r^{-1}\sup_{r\in\mathcal{R}(n)}\sup_{u\notin[0,\mathfrak{t}]^d}|K(u/r)| \le \|g\|_\infty\|K\|_\infty V_r^{-1}n^{-1}\\ &=: n^{-1}G_\infty(r) = n^{-1}G_\infty(\mathfrak{h}^{(d)}).\end{aligned}$$

$\blacksquare$

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